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Differential Calculus

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I. INTRODUCTION

These notes are based on the lectures delivered by the author to engineering students of London South Bank University over the period of 16 years. This is a University of widening participation, with students coming from many different countries, many of them not native English speakers. Most students have limited mathematical background and limited time both to revise the basics and to study new material. A system has been developed to assure efficient learning even under these challenging restrictions. The emphasis is on systematic presentation and explanation of basic abstract concepts. The technical jargon is reduced to the bare minimum.

Nothing gives a teacher a greater satisfaction than seeing a spark of understanding in the students' eyes and genuine pride and pleasure that follows such understanding. The author's belief that most people are capable of succeeding in - and therefore enjoying - the kind of mathematics that is taught at Universities has been confirmed many times by these subjective signs of success as well as genuine improvement in students' exam pass rates. Interestingly, no correlation had ever been found at the Department where the author worked between the students' qualification on entry and graduation.

The book owns a lot to the authors' students – too numerous to be named here - who talked to her at length about their difficulties and successes, e.g. see Appendix VII on Teaching Methodology. One former student has to be mentioned though – Richard Lunt – who put a lot of effort into making this book much more attractive than it would have been otherwise.

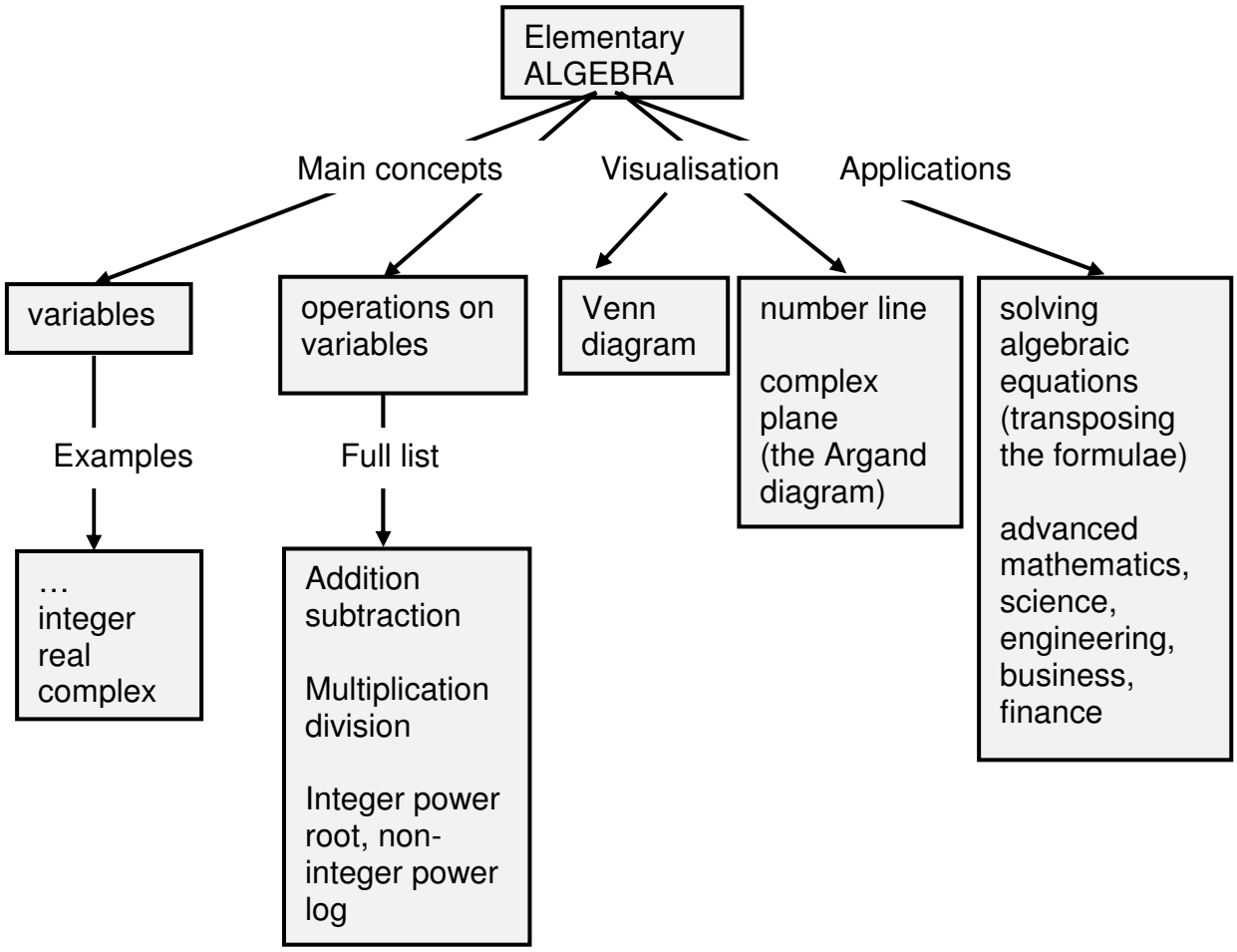
The author can be contacted through her website www.soundmathematics.com. All comments are welcome.

Good luck everyone!

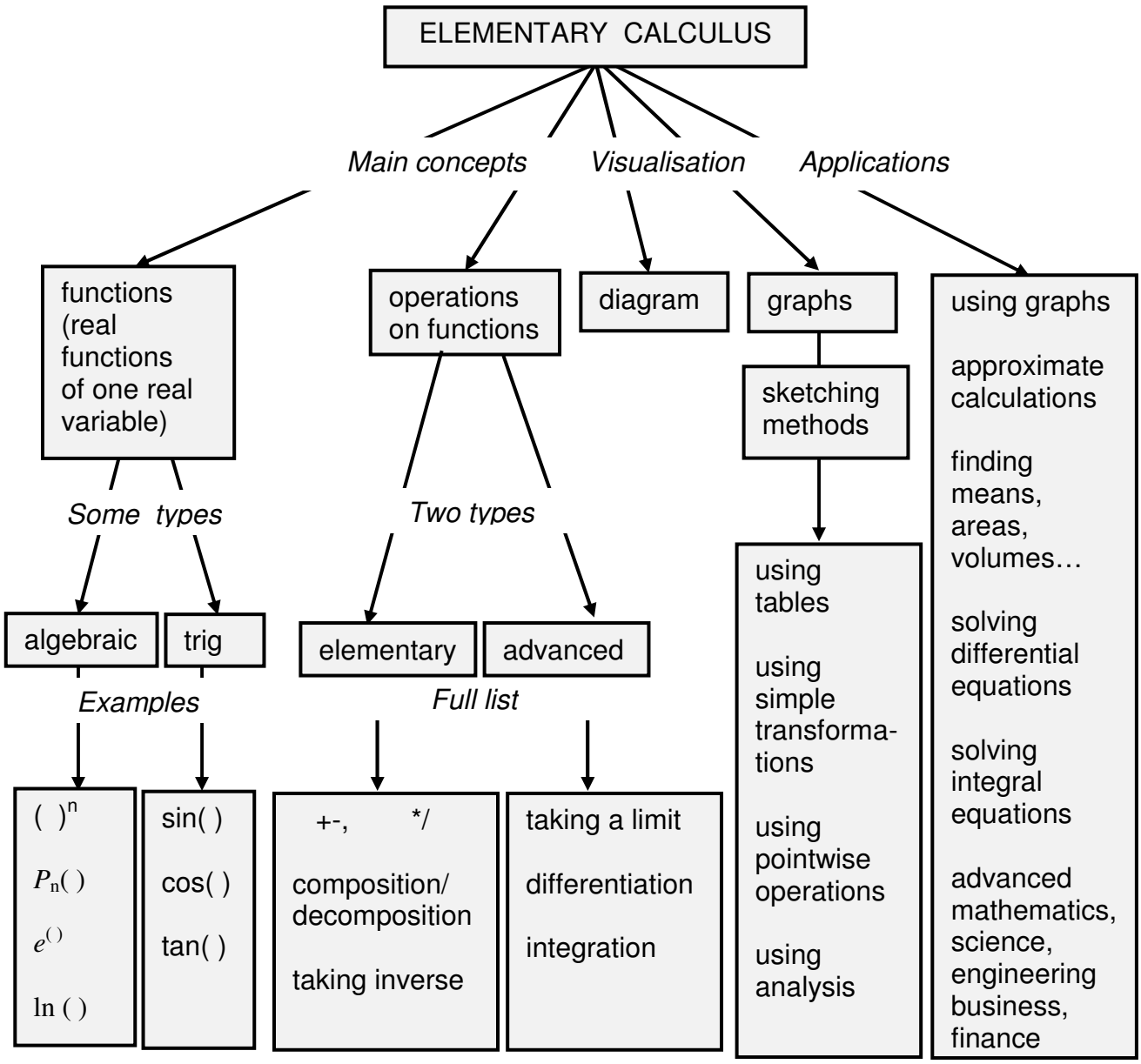
II. CONCEPT MAPS

Throughout when we first introduce a new **concept** (a technical word or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea.

One major topic is covered in this course, Differential Calculus. You can understand this topic best if you first study the Notes on Elementary Algebra and Functions. Here is a **concept map** of Elementary Algebra.



Differential Calculus is part of Elementary Calculus. Here is a concept map of Elementary Calculus. It is best to study it before studying any of the Calculus Lectures to understand where they are on the map. The better you see the big picture the faster you learn!



III. LECTURES

Lecture 13. CALCULUS: Sequences, Limits and Series

In Calculus we study **functions and operations on functions** (see Calculus Concept Map). Elementary operations on functions (algebraic operations and composition/decomposition) were covered in Lectures 4 - 7. We now introduce the first **advanced operation on functions** – **taking a limit**. We introduce it first for discrete functions called sequences.

13.1 Sequences

A **sequence** is an ordered set of numbers x_n, a **discrete function** $x(n)$, where the argument is an integer n. This means that a discrete function is a function whose domain is the set of all integers or else its subset. Such argument is called the **counter**.

There are different ways of writing up a sequence: $\{x_n\}$ or $x_1, x_2, x_3, ..., x_N$ (each symbol n, $1, 2, 3, \ldots, N$ here can also be called an **index** or subscript). A sequence can be **finite** (having a fixed number of elements) or **infinite** (whatever index you specify there is always an element in the sequence with a greater index). Traditionally sequence elements are called terms.

13.1.1. Defining a Sequence as Function of Counter

A sequence can be defined (described) by specifying a relationship between each element and its counter.

Examples:

1. Describe sequence $1, 2, 3, 4, ...$ using the above notation.

Solution

$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \ldots$

Question: What is the relationship between the counter and the sequence element?

Answer: $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \ldots$

$\Rightarrow \quad x_n = n$

2. Describe sequence $1, 4, 9, 16, ...$ using the above notation.

Solution

$x_1 = 1, x_2 = 4, x_3 = 9, ...$

Question: What is the relationship between the counter and the sequence element?

Answer: $x_1 = 1 = 1^2, \quad x_2 = 4 = 2^2, \quad x_3 = 9 = 3^2, ...$

$\Rightarrow x_n = n^2$

13.1.2. Visualising a Sequence as a Function of Counter

If a sequence is defined (described) by specifying a relationship between each element and its counter it can be visualised as a sequence of points on the number line (see figure 13.1) or else as a discrete graph (see figure 13.2).

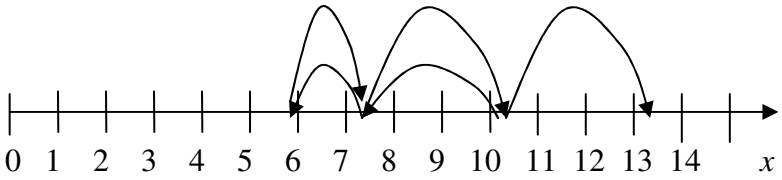


Figure 13.1. An example of a sequence x_n, $n = 0, 1, \ldots, 5$ visualised using the number line. The arrowed arcs are used to indicate the order of elements in the sequence.

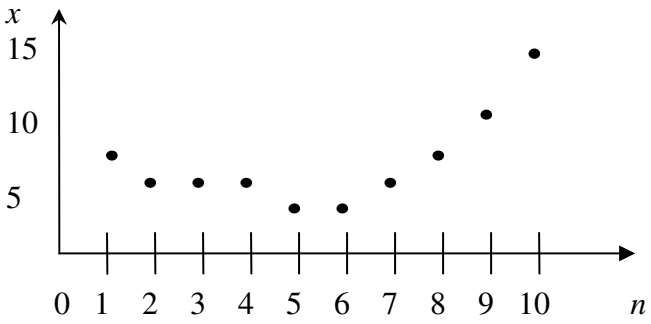


Figure 13.2. An example of a graph of sequence x_n, $n = 0, 1, \ldots, 10$.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

13.1.3 Applications of Sequences

A digital signal x (see figure 13.3 below) has the following characteristics:

1) it holds a fixed value for a specific length of time

2) it has sharp, abrupt changes

3) a preset number of values is allowed. **http://www.privateline.com/manual/threeA.html**

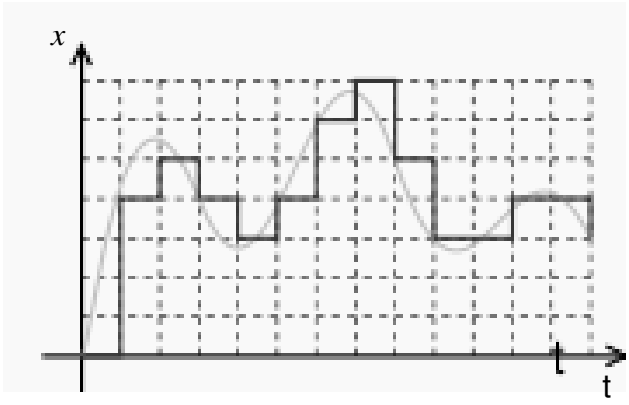


Figure 13.3. A typical digital signal. **http://www.privateline.com/manual/threeA.htm**

Therefore, digital signals are examples of sequences.

13.1.4. Defining a Sequence via a Recurrence Relationship

A sequence element may be defined in another way, via a **recurrence relation**

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_1).$$

where f may be a function of several arguments and not just one argument.

Thus, there are two ways to describe a sequence,
1. using a functional relation $x(n)$, which specifies how each sequence element is defined by its counter;

2. using a recurrence relation $x_{n+1} = f(x_n, x_{n-1}, ..., x_1)$. which specifies how each sequence element is defined by previous sequence element(s).

Examples:

1. Given a sequence $x_n = n^2$ we can change the functional description to a recurrence relation

$$x_{n+1} = (n+1)^2 = n^2 + 2n + 1$$
$$\Rightarrow x_{n+1} = x_n + 2\sqrt{x_n} + 1$$

When given such a recurrence the first element needs to be specified for us to be able to start evaluating other elements.

2. A **Fibonacci sequence**: 1, 1, 2, 3, 5, 8, 13, 21, ... can be described via a recurrence relation

$$x_{n+1} = x_n + x_{n-1}.$$

When given such a recurrence the first two elements have to be specified for us to be able to evaluate other elements. Let us check that this is a correct description:

Question: What are x_1 and x_2?
Answer: $x_1 = 1$ and $x_2 = 1$

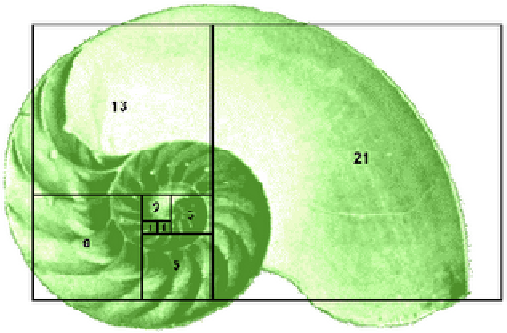
Question: Does x_3 satisfy the given recurrence relation and why?
Answer: Yes, because if the index is $n + 1 = 3$, then $n = 2$ and substitution into the relation gives us $x_3 = x_{2+1} = x_2 + x_{2-1} = x_2 + x_1 = 1 + 1 = 2$. Check!

Question: Does x_4 satisfy the given recurrence relation and why?
Answer: Yes, because if the index is $n + 1 = 4$, then $n = 3$ and substitution into the relation gives us $x_4 = x_{3+1} = x_3 + x_{3-1} = x_3 + x_2 = 2 + 1 = 3$. Check!

Question: Does x_5 satisfy the given recurrence relation and why?
Answer: Yes, because if the index is $n + 1 = 5$, then $n = 4$ and substitution into the relation gives us $x_5 = x_{4+1} = x_4 + x_{4-1} = x_4 + x_3 = 3 + 2 = 5$. Check!

Fibonacci sequences in nature

When superimposed over the image of a nautilus shell we can see a Fibonacci sequence in nature:



http://munmathinnature.blogspot.com/2007/03/fibonacci-numbers.html

Each of the small spirals of broccoli below follows the Fibonacci's sequence.



http://www.pdphoto.org/PictureDetail.php?mat=pdef&pg=8232

13.2 Limit of a sequence

Taking a limit of a sequence as n grows larger and larger without bounds is the first **advanced operation on functions** that we cover. In mathematics, instead of the phrase n *grows larger and larger without bounds* we use the shorthand $n \to \infty$ (verbalised as n tends to infinity). If it exists the outcome of applying this operation to a (discrete) function x_n is called $\lim x_n$ and is either a number or else $\pm\infty$ (either +**infinity** or -**infinity**). Sometimes instead of $\lim x_n$ we write $\lim_{n\to\infty} x_n$. However, usually, when description $n \to \infty$ is usually understood and not mentioned.

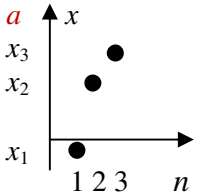
Note: ∞ is not a number but a symbol of a specific sequence behavior, $+\infty$ means that the sequence increases without bounds and $-\infty$ means that the sequence decreases without bounds - see the right column in the Table in Section 13.2.1 below.

13.2.1 Definition of a Limit of a Sequence

Examples of a finite limit $\lim_{(n\to\infty)} x_n = a$

(can also write $x_n \underset{(n\to\infty)}{\to} a$ **)**

Example 1: $x_n \to a-$



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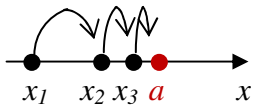
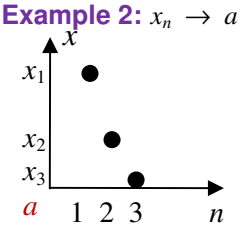


x_n gets closer and closer to a from below, maybe reaches it.

Example 2: $x_n \to a+$



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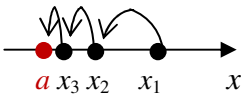
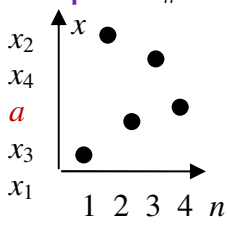


x_n gets closer and closer to a from above, maybe reaches it.

Example 3: $x_n \to$ a



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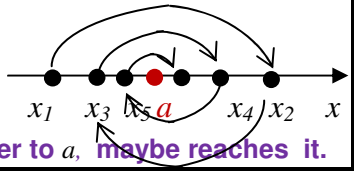
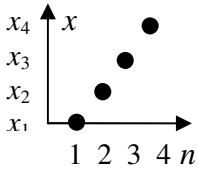


x_n gets closer and closer to a, maybe reaches it.

Examples of an infinite limit $\lim_{(n\to\infty)} x_n = \pm\infty$

(can also write $x_n \underset{(n\to\infty)}{\to} \pm\infty$ **)**

Example 1: $x_n \to +\infty$



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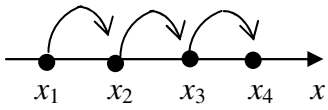
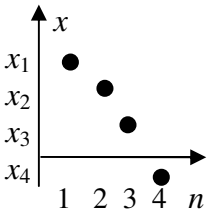


For any number M there exists $N : x_n > M$ for all $n \geq N$ (sequence increases without bounds)

Example 2: $x_n \to -\infty$



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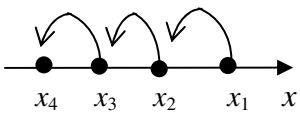


For any number M there exists $N : x_n < M$ for all $n \geq N$ (sequence decreases without bounds)

Examples: Sketch the sequence to find its limit:

1. $x_n = \frac{1}{n} \to 0$ **Verbalise:** as n increases, x_n tend (come closer and closer) to 0.
2. $x_n = -\frac{1}{n} \to 0$ **Verbalise:** as n increases, x_n tend (come closer and closer) to 0.
3. $x_n = \frac{(-1)^{n+1}}{n} \to 0$ **Verbalise:** as n increases, x_n tend (come closer and closer) to 0.
4. $x_n = n \to \infty$ **Verbalise:** as n increases, x_n tend to ∞ (increase without bounds).
5. $x_n = -n \to -\infty$ **Verbalise:** as n increases, x_n tend to $-\infty$ (decrease without bounds).

Sometimes, limits can be found using analytical considerations or else using the **Table**,

Rules and a **Decision Tree for Limits**:

13.2.2 Table of Limits

x_n	$\lim x_n$	
$1/n$	0	easy to find
$a^{1/n}, a>0$	1	
q^n, $\|q\| < 1$	0	the proof is rather involved ($e \approx 2.71$, **Neper's constant**, an irrational number).
$(1+\frac{1}{n})^n$	e	

If $\{x_n\}$ has a finite limit the sequence is said to **converge**.

13.2.3 Rules for (Properties of) Limits

1. The limit is uniqie.

2. $\lim (\alpha x_n + \beta y_n) = \alpha \lim x_n + \beta \lim y_n$, α, β – constant
Verbalise: limit of a sum is a sum of limits and any constant factor can be taken out – **linearity property.**

3. $\lim x_n y_n = \lim x_n \lim y_n$
Verbalise: limit of a product is a product of limits.

4. $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$, $\lim y_n \neq 0$
Verbalise: limit of a ratio is a ratio of limits.

Laws 2 - 4 give a general recipe for evaluating a limit of a composition of sequences – **instead of the sequence element substitute its limit.** That is, the Laws 2 – 4 state that if a limit exists the operation of taking the limit and any algebraic operation can be performed in an arbitrary order.

Optional

5. $x_n \geq y_n \Rightarrow \lim x_n \geq \lim y_n$ - **every element of one sequence is greater than or equal to the elment with the same index in the other sequence $\Rightarrow$ its limit is also greater than or equal to the limit of the other sequence.**

6. $cx_n \leq y_n \leq z_n \Rightarrow \lim x_n = \lim z_n \Rightarrow \lim y_n = \lim x_n$
a sequence squeezed between two others with the same limit has the same limit.

Example: $x_n = (-1)^{n+1} n$. $\lim x_n$?
Solution: $\lim x_n$ does not exist $\Leftarrow$ Rule 1

13.2.4 Indeterminate forms

Sometimes using the above rules leads to an **indeterminate form** (no obvious answer) like

$$\frac{0}{0}, \frac{\infty}{\infty}, \ 0\cdot\infty, \ \infty-\infty, \ 0^0, \ 1^\infty \ \textit{etc.} \ \text{-} \quad \textbf{MEMORISE}$$

Question: Why do these outcomes present no clear answer?

Answer: $\frac{0}{0}$ is indeterminate, because on the one hand, 0 divided by any number is 0 and on the other hand, division of any number by smaller and smaller numbers leads to larger and larger answers – tends to ∞ . $\frac{\infty}{\infty}$ is indeterminate, because on the one hand, ∞ divided by a number is ∞ and on the other hand, any number divided by ∞ (that is by larger and larger numbers) becomes smaller and smaller – tends to 0 *etc.*

When substitution leads to an indeterminate form we use tricks in an attempt to evaluate the limit or determine it to be non-existent.

Examples:

$\frac{0}{0}$ ↓ **use algebraic trick – FLIP RULE**

1. $x_n = \frac{1}{n^2}, \ y_n = \frac{1}{n} \Rightarrow \lim \frac{x_n}{y_n} = \lim \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim \frac{1}{n} = 0$

$\frac{0}{0}$ ↓ **use algebraic trick – FLIP RULE**

2. $x_n = \frac{1}{n^2}, \ y_n = \frac{1}{n} \Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim n = \infty$

3. $x_n = n, \ y_n = n^2$

$\infty - \infty$ ↓ **use algebraic trick - factorisation**

$\Rightarrow \lim(x_n - y_n) = \lim(n - n^2) = \lim n(1-n) = \lim n \lim(1-n) = \infty\cdot(-\infty) = -\infty$

4. $x_n = n^2, \ y_n = n^2 - 1$

$\frac{\infty}{\infty}$ ↓ **use algebraic trick – divide top and bottom by the highest power**

5. $\Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{n^2-1}{n^2} = \lim \frac{n^2 - 1 \cancel{\div n^2}}{n^2} = \lim \frac{1-\frac{1}{n^2}}{1} = 1 - 0 = 1$

13.2.5 Decision Tree for Limits

Decision Tree (a Flow Chart) for evaluating limits is given in figure 13.1. Start reading it from the very top and follow the arrows as dictated by your answers. The tricks required to find a limit can be very complicated. However, in standard undergraduate courses they involve only simple algebraic and trigonometric laws and formulae.

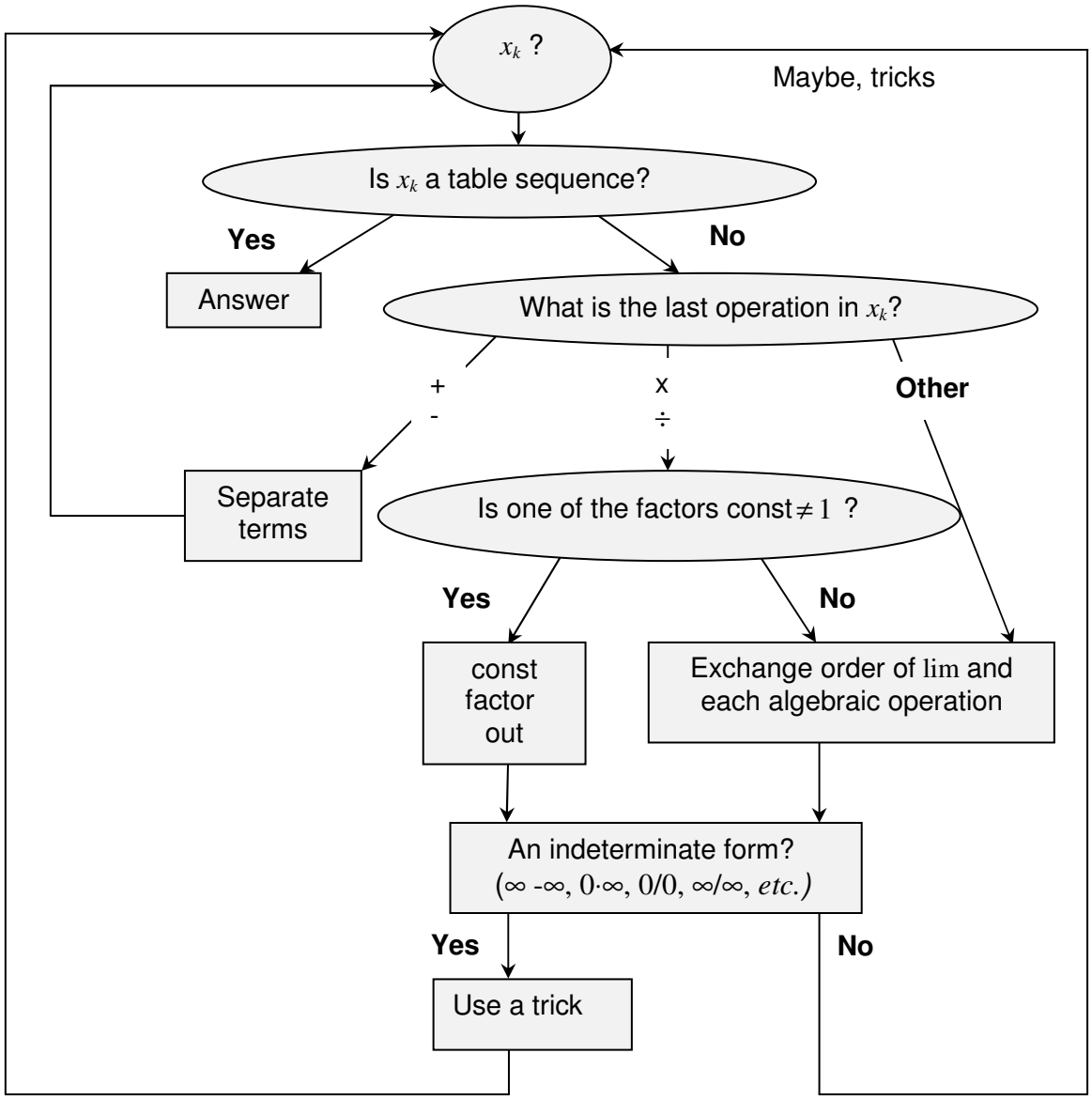


Figure 13.1. Decision Tree for Limits.

13.3 Series

A **series** is a sum of sequence elements,
Question: What is a sum?
Answer: An expression where the last operation is addition.

A series $x_1 + ... + x_N$ can be written using the shorthand S_N or $\sum_{n=1}^{N} x_n$. In the latter case we say that we use the **sigma notation**, because $\sum$ is a capital Greek letter called *sigma.*

13.3.1 Arithmetic Progression

An arithmetic progression is a sequence defined by the following recurrence relationship: $x_{n+1} = x_n + a$, where a – constant (with respect to n). a is called the **common difference.**

$$\Rightarrow S_N = x_1 + \ldots + x_N = N\frac{x_1 + x_N}{2}$$ **MEMORISE**

13.3.2 Geometric Progression

A geometric progression is a sequence defined by the following recurrence relationship: $x_{n+1} = ax_n$, where where a – constant (with respect to n). a is called the **common ratio.**

$$\Rightarrow S_N = \frac{x_1(1-a^N)}{1-a}$$

If $|a| < 1$, then $S_N = \frac{x_1}{1-a} - \frac{x_1 a^N}{1-a} \to \frac{x_1}{1-a} \Rightarrow \lim S_N = \frac{x_1}{1-a}$ **MEMORISE**

13.4 Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and TRIGONOMETRY**
- **Revise Lecture 11 and study Solutions to Exercises in Lecture 11 using the STUDY SKILLS Appendix**
- **Revise Lecture 12 using the STUDY SKILLS Appendix**
- **Study Lecture 13 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find the following limits:

a) $\lim \frac{n^2+n+1}{2n+1}$;

b) $\lim \frac{n^2+n+1}{2n^2+1}$;

c) $\lim \frac{n^2+n+1}{2n^3+1}$

Q2. Find the sum of a hundred terms of the arithmetic series with first term 1 and common difference 2.

Q3. A steel ball-bearing drops on to a smooth hard surface from a height h. The time to first impact is $T = \sqrt{\frac{2h}{g}}$, where g is the acceleration due to gravity. The times between successive bounces are $2eT, 2e^2T, 2e^3T, \ldots$, where e is the coefficient of restitution between the ball and the surface $(0 < e < 1)$. Find the total time taken up to the fifth bounce if $T = 1$ and $e = 0.1$.

Q4 (**advanced**). The irrational number π can be defined as the area of a unit circle (circle of radius 1). Thus, it can be defined as a limit of a sequence of known numbers a_n. The method used by Archimedes was to inscribe in the unit circle a sequence of regular polygons (that is, polygons with equal sides – see figure 8.1). As the number of sides increases the polygon "fills" the circle. For any given number n find the area a_n of the regular polygon inscribed in the unit circle. Show, by use of trigonometric identities

$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras theorem), that the area a_n satisfies the recurrence relation

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1 - \sqrt{\left[1 - \left(\frac{2a_n}{n}\right)^2\right]}, \quad n \geq 4.$$

Use trigonometry to show that $a_4 = 2$ and use the above recurrence relation to find a_8.

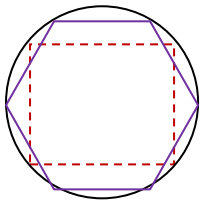


Figure 8.1. A circle with two inscribed regular polygons, one with four sides (a square) and one, with six (a hexagon).

Q5. Explain why $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$.

Lecture 14. DIFFERENTIAL CALCULUS: Limits, Continuity and Differentiation of Real Functions of One Real Variable

14.1 Limits

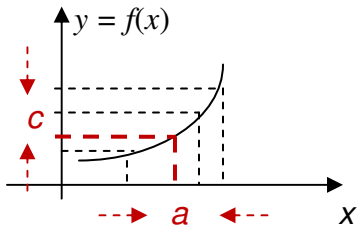
Taking a limit is **an advanced operation on functions**. The outcome of this operation is a number (more generally, a constant with respect to the independent variable). The operation involves finding a limiting value of the dependent variable when the independent variable approaches (tends to) a specified limiting value of its own or else to ∞ or ∞.

14.1.1 Definition of a Limit of a Real Function of One Real Variable

There are four possible situations when evaluating the limit of a function:

1. $\lim_{x \to a} f(x) = c$

Verbalise: as x tends to a finite limit a, $f(x)$ tends to a finite limit c.



2. $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$

Verbalise: as x tends (goes off) to infinity, f(x) tends to a finite limit c.

6. $\lim_{x \to \infty} f(x) = \infty$ or $-\infty$, $\lim_{x \to -\infty} f(x) = \infty$ or $-\infty$,

Verbalise: as x tends (goes off) to + or - infinity, $f(x)$ tends to + or - infinity.

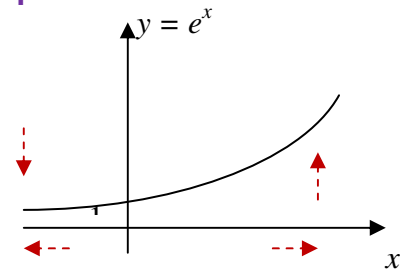
4. $\lim_{x \to a} f(x) = \infty$ or $-\infty$

Verbalise: as x tends to (approaches) a finite limit a, $f(x)$ tends to + or - infinity.

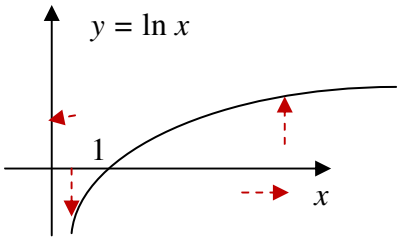
Note: symbols $\pm\infty$ are not numbers, they describe a specific function behaviour.

Examples:

1.



$\lim_{x \to -\infty} e^x = 0$ $\lim_{x \to \infty} e^x = \infty$

2. 

$\lim_{x\to 0+} \ln x = -\infty$ $\lim_{x\to\infty} \ln x = \infty$

14.1.2 Table of Limits of Functions

$f(x)$	$\lim_{x\to 0} f(x)$
$\frac{\sin x}{x}$	1
$\frac{e^x - 1}{x}$	1

} the proof is rather involved

14.1.3 Rules for (Properties of) Limits

1. The limit is unique (that is, there can be only one limiting value).
2. $\lim_{x\to a}[f_1 \begin{matrix}+\\-\\\times\\\div\end{matrix} f_2(x)] = \lim_{x\to a} f_1(x) \begin{matrix}+\\-\\\times\\\div\end{matrix} \lim_{x\to a} f_2(x)$ if all limits exist

Using rule 2 is equivalent to just substituting a for x. If substitution produces an indeterminate form tricks have to be used to establish whether the limit exists and if yes, find it.

Optional

3. $f_1(x) \le f_2(x) \le f_3(x)$ and $\lim_{x\to a} f_1(x) = \lim_{x\to a} f_3(x)$

$\Rightarrow \lim_{x\to a} f_2(x) = \lim_{x\to a} f_1(x)$

Examples:

1. $\lim_{x\to 1}(x^2 - 4x + 8) = 1^2 - 4\cdot 1 + 8 = 5$ **(instead of x we substituted its limiting value)**
2. $\lim_{x\to \pi/2} \sin x = 1$ **(instead of x we substituted its limiting value)**
3. $\lim_{x\to \pi/2} \cos x = 0$ **(instead of x we substituted its limiting value)**

4. $\lim_{x\to\pi/2}\tan x$ does not exist, but $\lim_{x\to\pi/2^{\pm}}\frac{\sin x}{\cos x}=\mp\infty$, which is a short-hand for two statements, $\lim_{x\to\pi/2+}\frac{\sin x}{\cos x}=-\infty$ and $\lim_{x\to\pi/2-}\frac{\sin x}{\cos x}=+\infty$

$\frac{0}{0}$

5. $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=4$

Verbalise: 0 over 0 indeterminate from – we find the limit using an algebraic trick.

Sometimes, limits can be found using analytical considerations or else using the **Table**, **Rules** and a **Decision Tree for Limits** (practically the same as the one for sequences).

14.1.4 Applications of the Concept of a Limit

1. Imagine a person walking over a landscape represented by the graph of $y = f(x)$. Her horizontal position is measured by the value of x, much like the position given by a map of the land or by a global positioning system. Her altitude is given by the coordinate y. She is walking towards the horizontal position given by $x = a$. As she does so, she notices that her altitude approaches L. If later asked to guess the altitude over $x = a$, she would then answer L, even if she had never actually reached that position.
http://en.wikipedia.org/wiki/Limit_of_a_function#Motivation
2. It is often important for engineers to assess how a measured quantity (current, voltage, density, load) behaves with time: does it grow without bounds (then its limit is ∞), tends to a specific value (then its limit is a finite number), oscillates (then it has no limit) *etc.*

14.1.5 Existence of a Limit of a Function

A limit of a function does not always exist.

Examples:

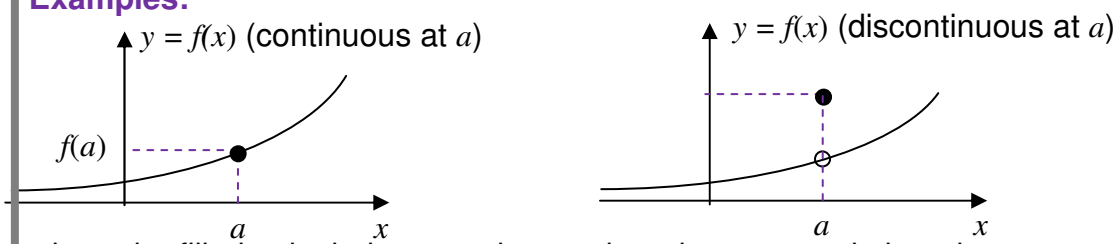
1. $\lim_{x\to\infty}\sin x$ does not exist.

2. $\lim_{x\to\infty}\sin\frac{1}{x}=0.$

14.2 Continuity of a function

A function $f(x)$ is said to be **continuous** at point a if $\lim_{x\to a} f(x)=f(a)$.

Examples:



where the filled point belongs to the graph and an empty circle – does not.

If a function is continuous at each point of an interval it is **continuous on** this **interval**. **Intuitively, a continuous function is a function for which, smaller and smaller changes in the input (argument or independent variable) result in smaller and smaller changes in the output (value of a function or dependent variable).** Otherwise, a function is said to be **discontinuous**.

If $f_1(x)$ and $f_2(x)$ are continuous on an interval, so is $f_1(x) \begin{matrix} + \\ - \\ \times \\ \div \end{matrix} f_2(x)$ (provided $f_2(x) \neq 0$ when dividing).

14.2.1 Applications of Continuous Functions

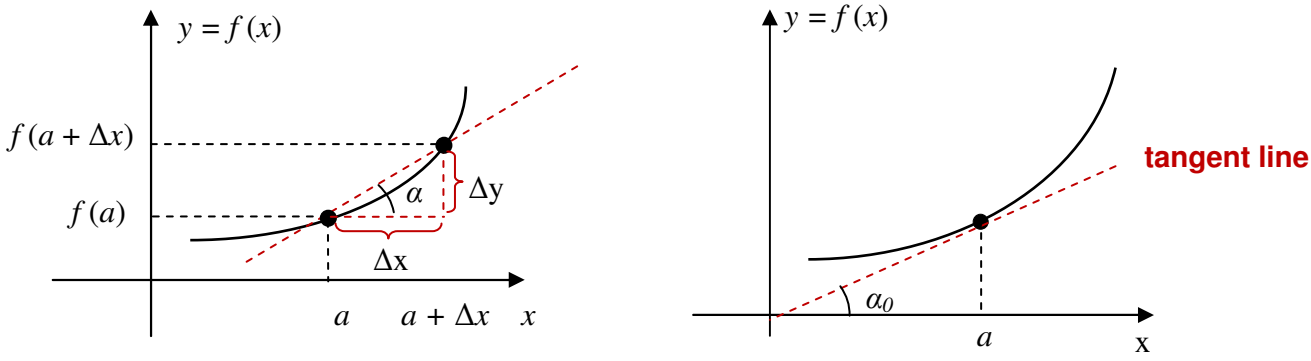
"An analogue signal is a continuous signal for which the time varying feature of the signal is a representation of some other time varying quantity... For example, in sound recording, fluctuations in air pressure (that is to say, sound) strike the diaphragm of a microphone which causes corresponding fluctuations in a voltage or the current in an electric circuit. The voltage or the current is said to be an 'analog' of the sound."
http://en.wikipedia.org/wiki/Analog_signal

14.3 Differentiation of a real function of one real variable

Differentiation is the second advanced operation on functions we cover. The outcome of this operation is a function, which is called **the derivative** of the original function.

14.3.1 A Derivative of a Function

A continuous function $f(x)$ is **differentiable** at $x = a$ if there exists $\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{0}{0}$



$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan\alpha = \tan\alpha_0 \equiv \frac{dy}{dx}(a) \equiv \frac{df}{dx}(a) \equiv f'(a)$ This limit is called a **derivative**.

Thus, we arrived at the following **definition of the derivative**:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

14.3.2 Geometrical Interpretation of a Derivative

A derivative of a function is a local slope (gradient) of the function at a point. More specifically, it is the slope of the line tangent to the graph of the function at this point.

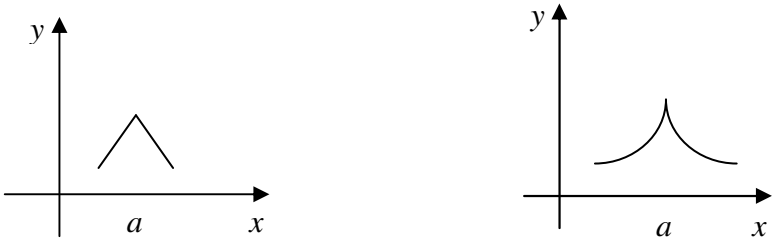
14.3.3 Mechanical Interpretation of a Derivative

A derivative of a function is a local rate (speed) of change of the function at a point.

14.3.4 Existence of a Derivative

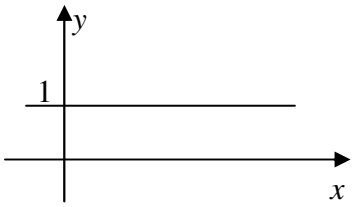
A derivative of a function does not always exist.

Here are graphs of functions which have no derivative at point a of their domain:



Examples:

1. Differentiate $y = c$ (constant)

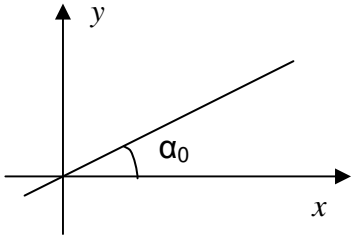


Answer: $\frac{dy}{dx} = 0$

a) Can differentiate using a graph: the slope is the same everywhere and is zero.
b) Can differentiate using the basic definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

2. Differentiate $y = c\,x$



Answer: $\frac{dy}{dx} = c$

a) Can differentiate using a graph: the slope is the same everywhere – c
b) Can differentiate using the basic definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c(x+\Delta x) - cx}{\Delta x} = c$$

14.4 A historical note

"When Newton and Leibniz first published their results on calculus in the 17th century, there was great controversy over which mathematician (and therefore which country) deserved credit. Newton derived his results first, but Leibniz published first. Newton claimed Leibniz stole ideas from his unpublished notes, which Newton had shared with a few members of the Royal Society. This controversy divided English-speaking mathematicians from continental mathematicians for many years, to the detriment of English mathematics. A careful examination of the papers of Leibniz and Newton shows that they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. Today, both Newton and Leibniz are given credit for developing calculus independently. It is Leibniz, however, who gave the new discipline its name. Newton called his calculus 'the science of fluxions' ". **http://en.wikipedia.org/wiki/Calculus**

14.5 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 12 and study Solutions to Exercises in Lecture 12 using the STUDY SKILLS Appendix**
- **Revise Lecture 13 using the STUDY SKILLS Appendix**
- **Study Lecture 14 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (using $\lim_{x\to 0}\frac{\sin x}{x}=1$ and trigonometric identities)

b) $\lim_{x\to 0}\frac{e^{x}-e^{-x}}{2}$

c) $\lim_{x\to 0}\frac{\sin^2 3x}{x^3}$

d) $\lim_{x\to 0}\frac{x^2-1}{2x^2+3}$

e) $\lim_{x\to \infty}\frac{x^2-1}{2x^2+3}$

Q2. Discuss continuity of $y = \tan x$.

Q3. Plot $y = e^{x}$. Differentiate this function approximately using the graph and plot the approximate derivative $\frac{d}{dx}e^{x}$.

Q4. Prove

a) $\frac{d}{dx}x^{n}=nx^{n-1}$ (Hint: use the Binomial Theorem – look it up in books or Google it)

b) $\frac{d}{dx}e^x = e^x$

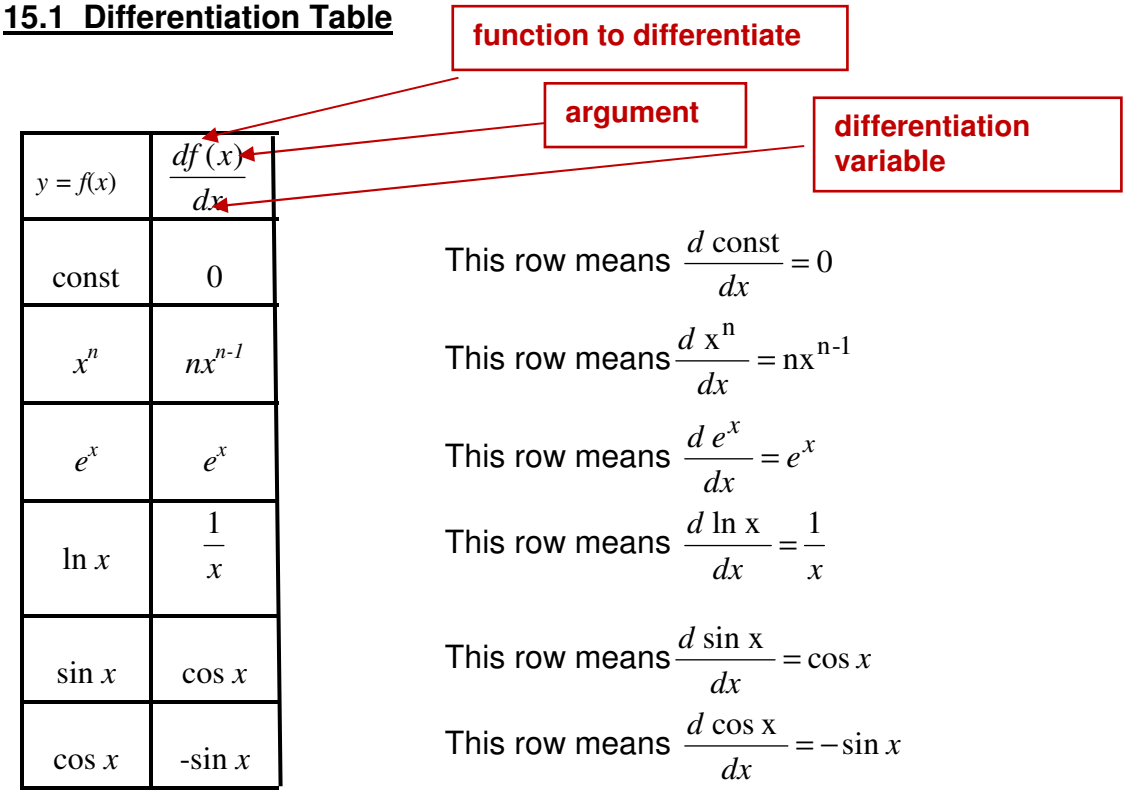
c) $\frac{d}{dx}\sin\ x = \cos\ x$

d) $\frac{\mathrm{d}}{\mathrm{dx}}\cos \mathrm{x} = -\sin \mathrm{x}$

Lecture 15. DIFFERENTIAL CALCULUS: Differentiation (ctd.)

Using **the first principles** (that is, the definition of a derivative) we can create a **table of derivatives of elementary functions**.

15.1 Differentiation Table



$y = f(x)$	$\frac{df(x)}{dx}$
const	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

This row means $\frac{d\ \text{const}}{dx} = 0$

This row means $\frac{d\ \text{x}^{\text{n}}}{dx} = \text{nx}^{\text{n-1}}$

This row means $\frac{d\ e^x}{dx} = e^x$

This row means $\frac{d\ \ln \text{x}}{dx} = \frac{1}{x}$

This row means $\frac{d\ \sin \text{x}}{dx} = \cos x$

This row means $\frac{d\ \cos \text{x}}{dx} = -\sin x$

Examples:

1. $\frac{\text{d}\cos\pi}{\text{dx}} = 0$

> **Differentiation variable?** x
> **Function to differentiate?** constant
> **It is a table function**

2. $\frac{\text{d}\,t^3}{\text{d}t} = 3t^2$

> **Differentiation variable?** t
> **Function to differentiate?** power 3
> **It is a table function of** t

3. $\frac{d \sin v}{dv} = \cos v$

Differentiation variable? v
Function to differentiate? $\sin(\ )$
It is a table function of v

4. $\frac{d(\ln t)^3}{d(\ln t)} = 3(\ln t)^2$

Differentiation variable? $\ln t$
Function to differentiate? power 3
It is a table function of $\ln t$

5. $\frac{d \cos e^v}{de^v} = -\sin e^v$

Differentiation variable? e^v
Function to differentiate? $\cos(\ )$
It is a table function of e^v

To differentiate combinations of elementary functions we can use **Differentiation Table, Dfferentiation Rules** and **Decision Tree** given in figure 15.1.

15.2 Differentiation Rules

1. $\frac{d\alpha f(x)}{dx} = \alpha \frac{df(x)}{dx}$ - **constant factor out rule**
2. $\frac{d[f(x)+g(x)]}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$ – **sum rule**

} **linearity property**

3. $\frac{d(f(x)g(x))}{dx} = \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x)$ – **product rule**
4. $\frac{d\ \frac{f(x)}{g(x)}}{dx} = \frac{\frac{df(x)}{dx} g(x) - \frac{dg(x)}{dx} f(x)}{g^2}$ – **quotient rule**
5. $\frac{d(f(g(x)))}{dx} = \frac{df(g)}{dg}\frac{dg}{dx}$ – **chain rule (decompose, differentiate, multiply).** Here $g = g(x)$

Note: These rules can be proven using the definition of the derivative as a limit and much simpler rules for limits.

Examples:

1. $\frac{d\ 2\ \sin\ x}{dx} = 2\cos\ x$

Differentiation variable? x
Function to differentiate? $2\sin x$
Last operation in this function? Multiplication
Is one of the factors constant? Yes
Apply the "constant factor out" rule

Constant factor $\alpha = 2$

Variable Factor $f(x) = \sin x$, $\frac{d\,f(x)}{dx} = \frac{d\ \sin\ x}{dx} = \cos x$

2. $\frac{d\ (\ \sin\ x\ + \cos\ x)}{dx} = \cos x - \sin x$

Differentiation variable? x
Function to differentiate? $3\sin x + 5\cos x$
Last operation in this function? addition
Apply the sum rule

---

f-line: 1st term $f(x) = \sin x$, $\frac{d\,f(x)}{dx} = \frac{d\ \sin\ x}{dx} = \cos x$

g-line: 2nd term $g(x) = \cos x$, $\frac{d\,g(x)}{dx} = \frac{d\ \cos\ x}{dx} = -\sin x$

3. $\frac{d\ (x^2 + \ 3x\ + 2\ )}{dx} = \frac{d\ x^2}{dx} + \frac{d\ 3x}{dx} + \frac{d\ 2}{dx} = 2x + 3$

Differentiation variable? x
Function to differentiate? $x^2 + 3x + 2$
Last operation in this function? Addition
Apply the sum rule
Go over the same reasoning for each term

4. $\frac{d\,(x\ \sin\ x)}{dx} = \sin\ x + x\ \cos\ x$

Last operation in this function? Multiplication
Is one of the factors constant? No
Apply the product rule

f-line: 1st factor $f(x) = x$, $\frac{d\,f(x)}{dx} = \frac{d\,x}{dx} = 1$

g-line: 2nd factor $g(x) = \sin x$, $\frac{d\,g(x)}{dx} = \frac{d\ \sin\ x}{dx} = \cos x$

5. $\frac{d\,(x\ \ln\ x)}{dx} = \ln\ x + 1$ (using the product rule)

f-line: 1st factor $f(x) = x$, $\frac{df\,(x)}{dx} = \frac{dx}{dx} = 1$

g-line: 2nd factor $g(x) = \ln x$, $\frac{dg\,(x)}{dx} = \frac{d\,\ln x}{dx} = \frac{1}{x}$

6. $\frac{d\,(\ \frac{\sin\ x}{x}\ )}{dx} = \frac{x\ \cos\ x\ -\ \sin\ x}{x^2}$ (using the quotient rule)

f-line: Numerator $f(x) = \sin x$, $\frac{d\,f(x)}{dx} = \frac{d\ \sin\ x}{dx} = \cos x$

g-line: Denominator $g(x) = x$, $\frac{d\,g(x)}{dx} = \frac{dx}{dx} = 1$

7. $\frac{d\left(\frac{\ln x}{x}\right)}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

Differentiation variable? x

f(x) ? $\frac{\ln x}{x}$

Last operation in this function? Division

Apply which rule? Quotient rule

---

f-line: Numerator $f(x)=\ln x,\ \frac{df(x)}{dx} = \frac{d\ \ln x}{dx} = \frac{1}{x}$

g-line: Denominator $g(x)=x,\ \frac{dg(x)}{dx} = \frac{dx}{dx} = 1$

8. $\frac{d\ \sin\ 2x}{dx} = 2\cos 2x$

Differentiation variable? x

f(x)? $\sin 2x$

Last operation in this function? $\sin(\ )$

Its argument? $2\ x$

Apply which rule? The chain rule

---

f-line: Last operation $f()=\sin(\ ),$

$\frac{df(x)}{dx} = \frac{d\ \sin(g)}{dg} = \cos g = \cos 2x$

g-line: Its argument $g(x)=2x,\ \frac{dg(x)}{dx} = \frac{d\ 2x}{dx} = 2$

9. $\frac{d \ln 3x}{dx} = 3\frac{1}{3x} = \frac{1}{x}$

Differentiation variable? x
Function to differentiate? $\ln 3x$
Last operation in this function? $\ln(\ )$
Its argument? $3x$
Apply the chain rule

f-line: Last operation $f(\ ) = \ln(\ ),\ \frac{df(x)}{dx} = \frac{d \ln(g)}{dg} = \frac{1}{g} = \frac{1}{3x}$

g-line: Its argument $g(x) = 3x,\ \frac{d\,g(x)}{dx} = \frac{d\,3x}{dx} = 3$

15.3 Decision Tree for Differentiation

Generally, to differentiate a function we use the **Decision Tree for Differentiation** (see figure 15.1 below) that allows us to decide whether to use the Differentiation Table (containing derivatives of elementary functions) straight away or first use Differentiation Rules (on how to differentiate combinations of elementary functions). Once the rule is applied, even if the problem is not solved, it is reduced to one or several simpler problems which have to be treated using the Decision Tree again.

To use the Decision Tree start at the top and follow the arrows which are associated with the correct answers (if any). Use it as a formula, substituting your differentiation variable for x and your function for $f(x)$.

15.4 The higher order derivatives

If a derivative of $f(x)$ is a differentiable function its derivative is called **the second derivative** of $f(x)$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

If the second derivativeof $f(x)$ is differentiable its derivative is called **the third derivative** of $f(x)$,

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \quad \textit{etc.}$$

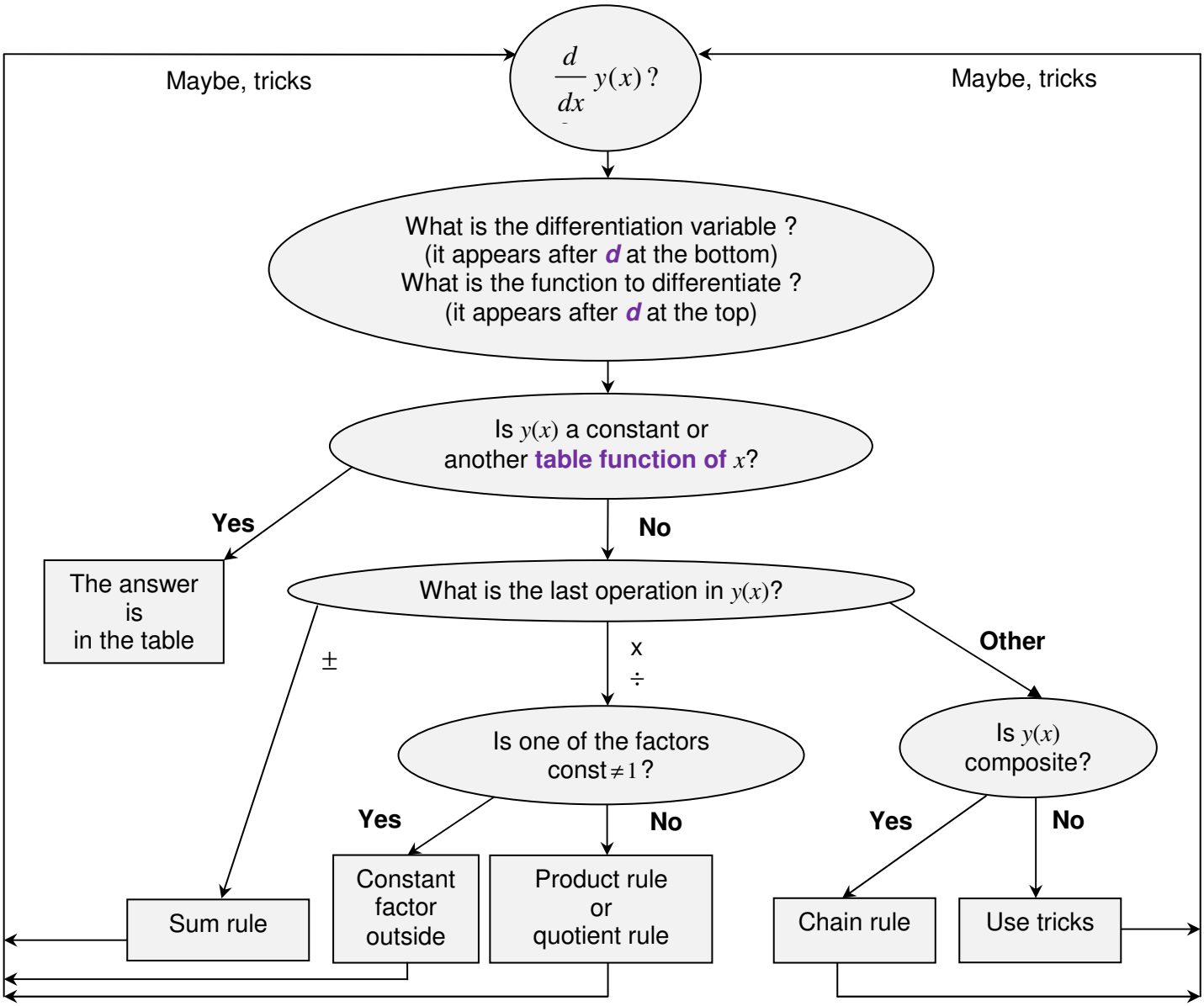


Figure 15.1. Decision Tree for Differentiation.

15.5 The partial derivatives

If we consider a function of say, two variables (two arguments) $f(x,y)$ its **partial derivatives** with respect to x and y are denoted by $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$, respectively.

15.6 Applications of differentiation

Later you will find that all simple engineering systems that you are going to study are described by differential equations that relate values of measured variables, such as displacement, angle or rotation, concentration, current or voltage to their first or second derivatives. You will be also shown that derivatives and limits are helpful in sketching functions and thus illustrating or even predicting behaviour of the above measurable variables. Finally you will find out how

derivatives can help to approximate functions, e.g. how your calculators calculate $\sin, \cos$ values *etc.*

15.7 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 4 and Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Lecture 13 and study Solutions to Exercises in Lecture 13 using the STUDY SKILLS Appendix**
- **Revise Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 15 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find a) $\dfrac{d}{d\sqrt{t^2+1}} e^{\sqrt{t^2+1}}$

b) $\dfrac{d}{d\sqrt{t^2+1}} e^{\sqrt{x^2+1}}$

Q2. Find the derivatives

a) $\dfrac{d}{du}(u^2+1)\sin u$

b) $\dfrac{d}{du}\dfrac{u^2+1}{\sin u}$

Q3. Find the derivatives

a) $\dfrac{d\cos\sqrt{x^2+1}}{dx}$

b) $\dfrac{d\cos\sqrt{x^2+1}}{dy}$

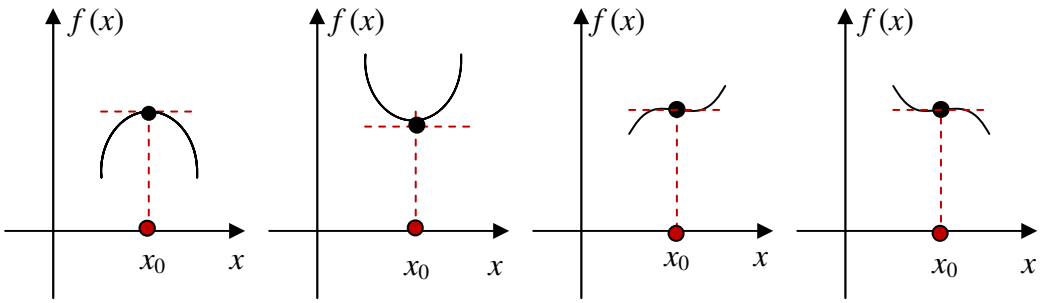
Q4. Differentiate y with respect to t if $y = (t+\sqrt{t^2+a^2})^n$.

Lecture 16. DIFFERENTIAL CALCUUS: Sketching Graphs Using Analysis

In general, any composite function can be sketched using our knowledge of limits and derivatives. In this Lecture we show how.

16.1 Stationary points

A **stationary point** is a point x_0 in the function domain where its derivative $f'(x_0) = 0$.



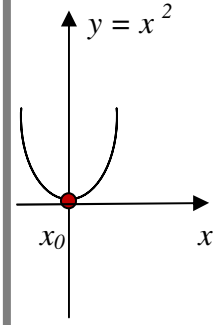
Examples: Sketch the following functions and find their stationary ponts x_0 :

1. $y = x^2$

$\frac{dy}{dx} = 0$

$2x = 0$

$x_0 = 0$

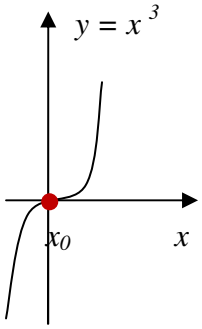


2. $y = x^3$

$\frac{dy}{dx} = 0$

$3x^2 = 0$

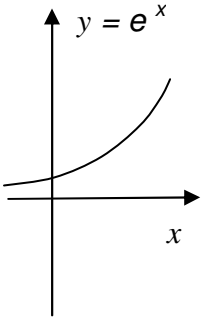
$x_0 = 0$



3. $y = e^x$

$\frac{dy}{dx} = e^x \neq 0$

No stationary points



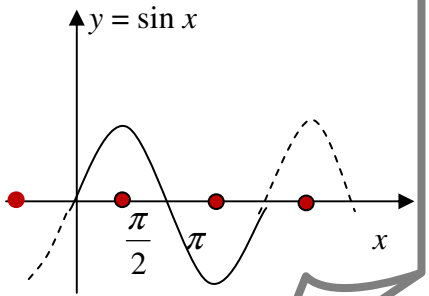
4. $y = \sin x$

$\frac{dy}{dx} = \cos x = 0$

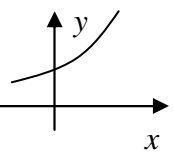
$x_0 = \frac{\pi}{2} n,$

n – odd integer

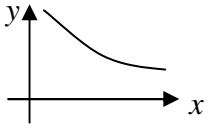
($n = 2m + 1$, m – integer)



16.2 Increasing and decreasing functions



If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **increasing** on I.

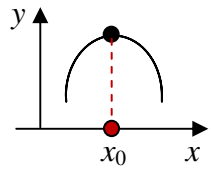


If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **decreasing** on I.

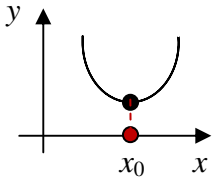
Examples: Functions $y = e^x$ and $y = \ln x$ are increasing everywhere; functions $y = |x|$ and $y = x^2$ are decreasing for $x < 0$ and increasing for $x > 0$.

16.3 Maxima and minima

If derivative (local slope) $f'(x) > 0$ to the left of x_0 and function $f'(x) < 0$ to the right of x_0, then function $f(x_0)$ is said to have a **maximum** at x_0.



If derivative (local slope) $f'(x) < 0$ to the left of x_0 and function $f'(x) > 0$ to the right of x_0, then function $f(x_0)$ is said to have a **minimum** at x_0.



Examples: $y = x^2$ has minimum at 0, x^3 has no minimum or maximum, $\sin x$ has maxima at $x = \frac{\pi}{2} m$ when m is odd and minima when m is even.

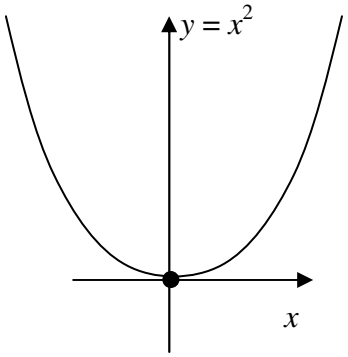
Examples:

1. Sketch $y = x^2 + 3x + 1$ by completing the square

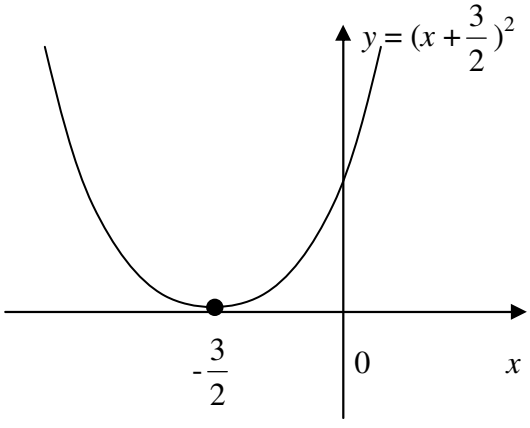
Solution:

Completing the square, $y = (x + \frac{3}{2})^2 - \frac{9}{4} + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$.

Step 1. Using the recipe for simple transformations, we first drop all constant factors and terms and sketch the basic shape of the function that remains:

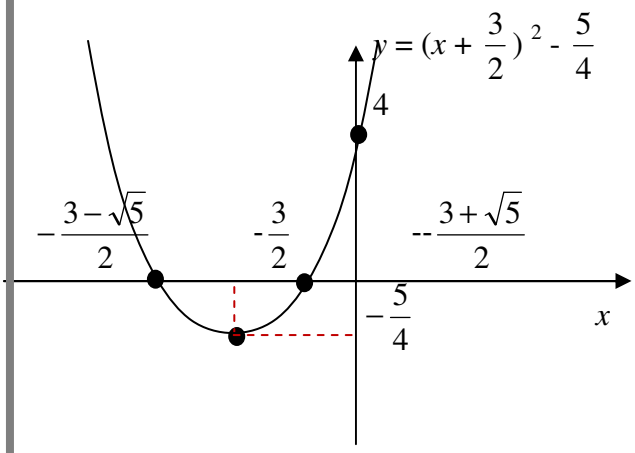


Step 2. We then reintroduce the constant factors and terms that appeared in the original functional equation one by one, in Order of Operations and sketch the resulting functions underneath one another:



$+\frac{3}{2}$ – constant term, first operation

$\Rightarrow$ translation wrt x-axis by $-\frac{3}{2}$



$-\frac{5}{4}$ – constant term, first operation

$\Rightarrow$ translation wrt y-axis by $-\frac{5}{4}$

Step 3.

Intersection with the y-axis: $x = 0,\ y = 4$

Intersection with the x-axis: $y = 0,\ x_{1,2} = -\frac{3 \pm \sqrt{9-4}}{2} = -\frac{3 \pm \sqrt{5}}{2} \approx -\frac{3 \pm 2.1}{2} \approx -2\frac{1}{2}, -\frac{1}{2}$

2. Sketch $y = x^2 + 3x + 1$ by analysis.

Solution

Step 1. **Intersection with the y-axis:** $x = 0,\ y = 4$

Intersection with the x-axis: $y = 0,\ x_{1,2} = -\frac{3 \pm \sqrt{9-4}}{2} = -\frac{3 \pm \sqrt{5}}{2} \approx -\frac{3 \pm 2.1}{2}$

Step 2. Find and sketch the first derivative

$$y' = 2x + 3$$

a) Find where the first derivative is zero: $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

$\Rightarrow$ Stationary point of the function is $x = -\frac{3}{2}$

b) Find where the first derivative is positive and where it is negative:

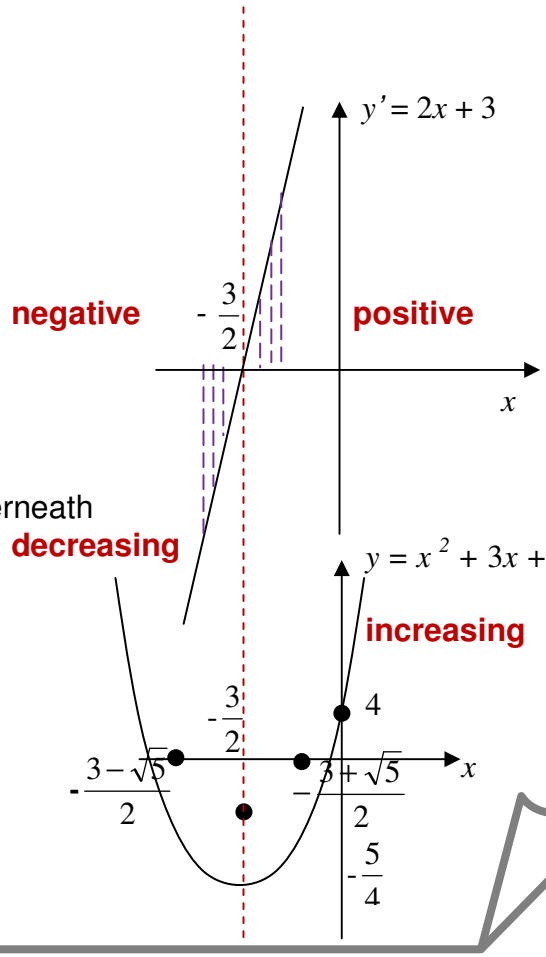
Can do so by solving the inequalities

$2x + 3 > 0 \quad (y' > 0)$ for $x > -\frac{3}{2}$

$2x + 3 < 0 \quad (y' < 0)$ for $x < -\frac{3}{2}$

Note: inequalities can be solved the same way as equations but when multiplying by negative factors the inequality sign should be reversed.

or graphically



Step 3: Sketch the function y directly underneath the first derivative y'

Step 4: Find the minimum function value

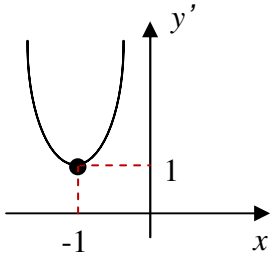
$$y(-\frac{3}{2}) = \frac{9}{4} - \frac{9}{2} + 1 = \frac{9-18+4}{4} = -\frac{5}{4}$$

3. Sketch $y = x^3 + 3x^2 + 4x + 1$ by analysis

Solution

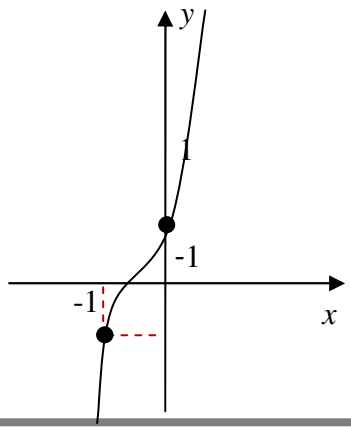
Step 1. Find and sketch the first derivative y'

$$y' = 3x^2 + 6x + 4 = 3(x^2 + 2x) + 4 = 3[(x+1)^2 - 1] + 4 = 3(x+1)^2 + 1 > 0$$



$y' > 0$ everywhere

Step 2. Sketch the function y underneath



y is increasing everywhere

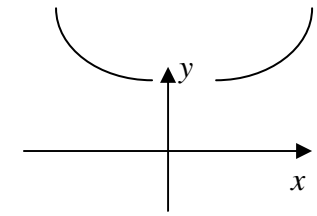
Special points: $y(0) = 1$

$y(-1) = -1$ (not that special – but easy to find)

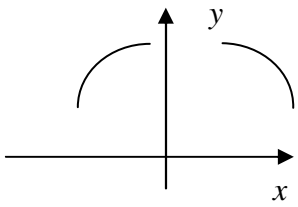
Optional

16.5 Convex and concave functions

If the second derivative $f''(x) > 0$ on interval I
the first derivative $f'(x)$ is increasing on I,
and function $f(x)$ is said to be **convex** on interval I.



If the second derivative $f'(x) < 0$ on interval I
the first derivative $f'(x)$ is decreasing on I
and function $f(x)$ is said to be **concave** on interval I.



Examples: establish the regions of convexity of the following elementary functions:

$y = x^2$

$y' = 2x$

$y'' = 2 > 0$

$\Rightarrow y = x^2$ is convex everywhere

$y = x^3$

$y' = 3x^2$

$y'' = 6x \begin{cases} > 0 \text{ if } x > 0 \\ < 0 \text{ if } x < 0 \end{cases}$

$\Rightarrow y = x^3$ is $\begin{cases} \text{convex if } x > 0 \\ \text{concave if } x < 0 \end{cases}$

$y = e^x$

$y' = e^x$

$y'' = e^x > 0$

$\Rightarrow y = e^x$ is convex everywhere

16.6 Inflexion points

If the second derivative $f''(x_0) = 0$ and $f''(x)$ changes sign at the stationary point x_0, then the stationary point x_0 is called **an inflexion point.**

Example: $y = x^3$ has an inflexion point at $x = 0$.

Thus, there is another way of determining whether the stationary point is at a maximum or minimum:

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the function is convex in the vicinity of x_0 and **x_0 is at a minimum.**

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the function is concave in the vicinity of x_0 and **x_0 is at a maximum.**

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then *further* **checks are required.**

16.7 Sketching rational functions using analysis

We illustrate the general principles of sketching by analysis by applying it to rational functions.

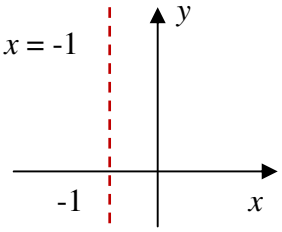
A **rational function** $R(x)$ is

$$R(x) = \frac{P(x)}{Q(x)}, \text{ where } P \text{ and } Q \text{ are polynomials.}$$

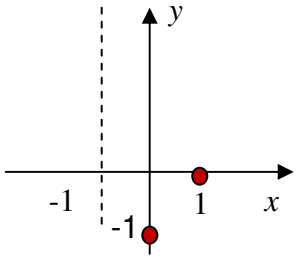
Examples: 1. Sketch the rational function $y = \dfrac{x-1}{x+1}$

Solution

Step 1. Domain: $x \neq -1$ ($x = -1$ is called **a pole**, the dashed line $x = -1$ is called **a vertical asymptote**)



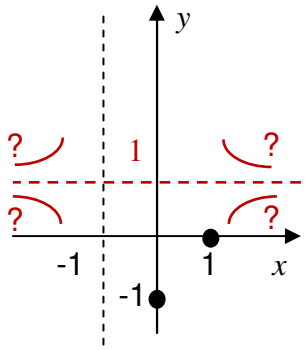
Step 2. Intersection with the y-axis: $x = 0$, $y = -1$
Intersection with the x-axis: $y = 0$, $x = 1$



Step 3. Behaviour at the domain boundaries:

$x \to \infty, y \to 1$
$x \to -\infty, y \to 1$

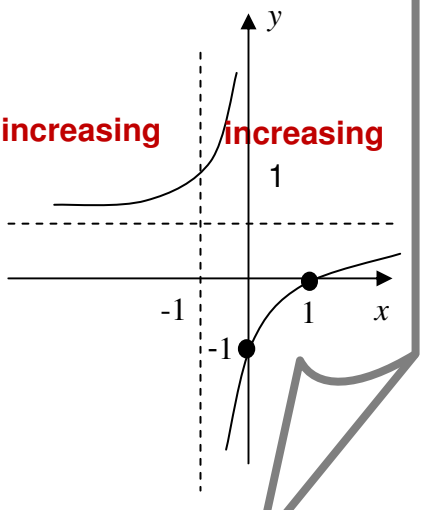
The dashed line $y=1$ is called **a horizontal asymptote**. We know that the function approaches it when $|x|$ grows large but we do not know how.



Step 4. Find the first derivative:

$$\frac{dy}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$$

$\Rightarrow y$ is increasing everywhere



2) Sketch $y = \dfrac{x-1}{x^2-4x+4}$

Solution

Step 1. Domain: $x \neq 2$ (x=2 is a pole, the lines is a vertical asymptote)

Step 2. Intersection with the y-axis: $x = 0,\ y = -\frac{1}{4}$

Intersection with the x-axis: $y = 0,\ x = 1$

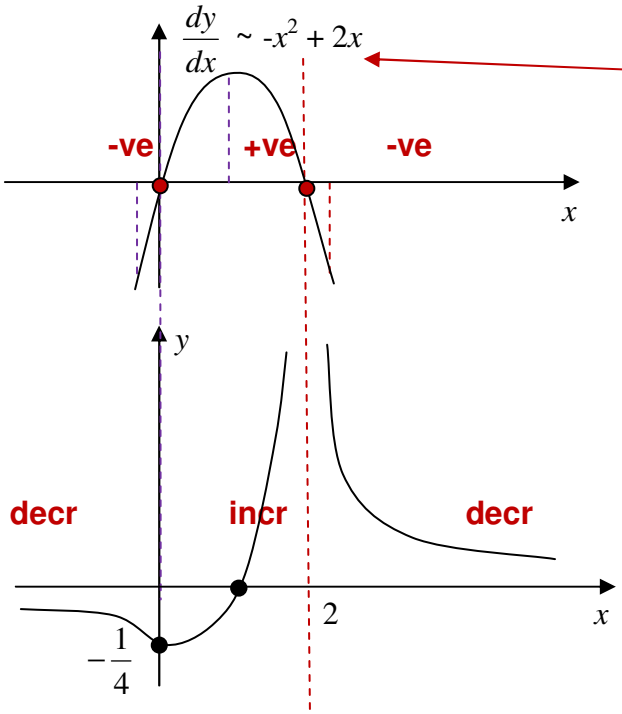
Step 3. Behaviour at the domain boundaries

$x \to \pm\infty,\ y \to \pm 0$

The line $y = 0$ is a **horizontal asymptote**

Step 4. Find the first derivative:

$$\frac{dy}{dx} = -\frac{x}{(x-2)^3} = \frac{-x^2+2x}{(x-2)^4} \begin{cases} = 0 & \text{at } x = 0 \\ > 0\ (y \text{ is increasing}) & \text{for } 0 < x < 2 \\ < 0\ (y \text{ is decreasing}) & \text{for } x < 0 \ \&\ x > 2 \end{cases}$$



This is not a derivative but it is 0, positive and negative where $\frac{dy}{dx}$ is – since $(x-2)^4 > 0$ when $x \neq 2$.

The sign ~ means "behaves as".

Note that $x = 2$ turns the numerator $-x^2 + 2x$ into zero but $\frac{dy}{dx}$ is not zero at this point. The "phantom" zero appears because we multiplied both numerator and denominator of $\frac{dy}{dx}$ by $(x - 2)$. This was a trick performed to make sure that the denominator is never negative.

Optional

Sketch $y = \dfrac{x^2+4x+2}{x+1}$ - **improper rational function**

polynomial degree 2 (numerator) / **polynomial degree** 1 (denominator)

Difference in degrees = 1 $\Rightarrow$ the whole part is a **polynomial degree** 1

$$\frac{x^2+4x+2}{x+1} = Ax+B + \frac{C}{x+1}$$

$\frac{x^2+4x+2}{x+1}$: **polynomial degree** 0

$Ax+B$: **polynomial degree** 1

To find A, B and C use

1. **long division** or
2. **partial fractions method**

$$\frac{x^2+4x+2}{x+1} = Ax + B + \frac{C}{x+1} \quad / \cdot (x+1)$$

$$x^2 + 4x + 2 = Ax(x+1) + B(x+1) + C$$

True for all x $\Rightarrow$ can choose any convenient value of x

1. Choose values that "kill" terms $x = -1$: $1 - 4 + 2 = C$, $C = -1$
   $x = 0$: $2 = B + C$, $2 = B - 1$, $B = 3$
2. Equate coefficients of the highest power: $1 = A$

$$y = x + 3 - \frac{1}{x+1}$$

Now we have to sketch $y = x + 3 - \dfrac{1}{x+1}$

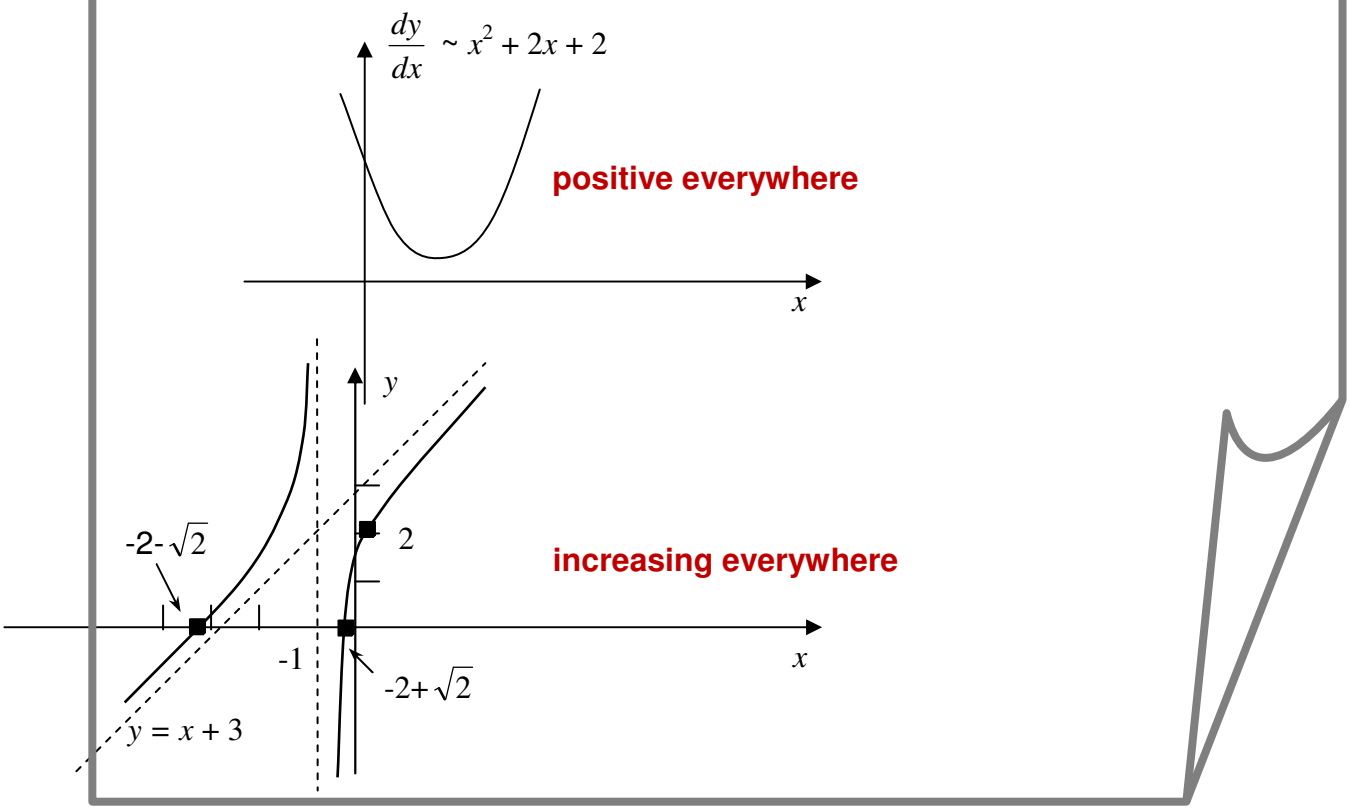
Step 1. Domain: $x \neq -1$ ($x = -1$ is a pole)

Step 2. **Intersection with the y-axis:** $x = 0$, $y = 2$
Intersection with the x-axis: $y = 0$, $x^2 + 4x + 2 = 0$, $x_{1,2} = -2 \pm \sqrt{2}$

Step 3. Behaviour at the domain boundaries:
$x \to \pm\infty$, $y \to x + 3$ – **an oblique asymptote** (typical for improper rational fractions)

Step 4. Find the first derivative:

$\frac{dy}{dx} = \frac{x^2 + 2x + 2}{(x+1)^2} > 0$ because $x^2 + 2x + 2 = 0$ at $x_{1,2} = -1 \pm j$ (no real roots)



16.8 Applications of rational functions and their graphs

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. Any such transfer function is a rational function of one complex variable. Sketching magnitudes and phases of these functions helps design efficient control systems.

16.9 Instructions for self-study

- **Revise Lecture 9 and study Solutions to Exercises in Lecture 9 (sketching by simple transformations)**
- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lectures 13 - 15 (limits and differentiation) and study Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 16 using the STUDY SKILLS Summary**
- **Do the following exercises:**

Q1. Sketch using analysis

1. $y = -3\,(x^3 - 3x^2 + x)$

2. $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Q2. Sketch $y = \frac{x^2 - 5x + 6}{x^2 + 1}$

Q3. Sketch $y = u(x + 2)\ \frac{x^2 - 5x + 6}{x^2 + 1}$

Q4. Sketch $y = \frac{x^2 - 5x + 6}{x + 1}$

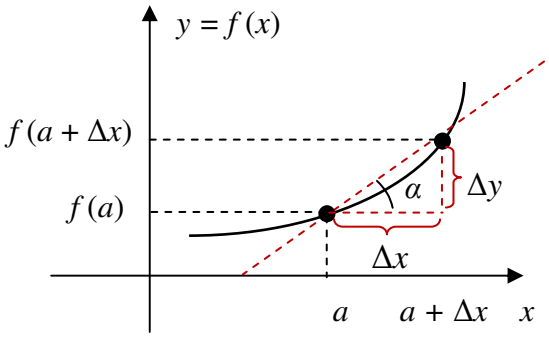
Lecture 17. Application of DIFFERENTIAL CALCULUS to Approximation of Functions: the Taylor and Maclaurin Series

17.1 Approximating a real function of a real variable using its first derivative

Let us discuss how we can **approximate** the value of function $f(x)$ at $a+\Delta x$ if we know its value at a. It is clear from the figure below that

$$f(a+\Delta x) \approx f(a) + \Delta y .$$

How can we approximate Δy? To see this, let us revise the definition of the first derivative given in Lecture 13.



Question: What is the geometrical interpretation of derivative?
Answer: It is a local slope, $f'(x) = \tan\alpha$
Question: How can we find its approximate value?
Answer: $\tan\alpha \approx \dfrac{\Delta y}{\Delta x}$ when Δx is small.
Question: How to solve this approximate equality for Δy?
Answer: Multiplying both sides by Δx we can see that $\Delta y \approx \tan\alpha\,\Delta x = f'(x)\Delta x$

$$\Rightarrow f(a+\Delta x) \approx f(a) + f'(a)\,\Delta x$$

Verbalise: The function value near a **approximately equals** the function value at a + derivative at a times distance to a.

This is an approximation *linear in* Δx (in the vicinity of x the graph of $f(x)$ looks like a *straight line*). Sometimes we need approximations based on a quadratic, cubic … n-*th* degree polynomial. Before showing how to find such approximations we introduce a new way of representing a polynomial.

17.2 The Maclaurin polynomials

Question: What is a polynomial?
Answer: A sum of products of constant coefficients with whole powers of the argument.

Question: Conventionally we write the first term of the polynomial $P_n(x)$ as $a_n x^n$. All coefficients are constant with respect to x **and are described by the letter** a. What will be the next term and the next and the next…?
Answer: The exponents decrease by 1 and so do subscripts on the corresponding coefficients.

Any polynomial can be re-written in another form as

the Maclaurin polynomial: $P_n(x) = P_n(0) + P_n'(0)x + \dfrac{P_n''(0)}{2!}x^2 + \dfrac{P_n'''(0)}{3!}x^3 + \ldots \dfrac{P_n^{(n)}(0)}{n!}x^n$

where n **factorial** $n! = 1\times 2\times 3\times 4\times \ldots \times n$.

We can show this using the following steps:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 \quad \Rightarrow \quad P_n(0) = a_0 \quad \Rightarrow \quad a_0 = P_n(0)$$

$$P_n'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + 3a_3 x^2 + 2a_2 x + a_1 \quad \Rightarrow \quad P_n'(0) = a_1 \quad \Rightarrow \quad a_1 = P_n'(0)$$

$$P_n''(x) = a_n n(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 x + 2a_2 \Rightarrow P_n''(0) = 2a_2 \Rightarrow a_2 = \frac{P_n''(0)}{2}$$

$$P_n'''(x) = a_n n(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 \quad \Rightarrow P_n'''(0) = 2\cdot 3\ a_3 \Rightarrow a_3 = \frac{P_n'''(0)}{3!}$$

… …

$$P_n^{(n)}(x) = a_n n(n-1)(n-2)....[n-(n-1)]x^{n-n} \quad \Rightarrow P_n^{(n)}(0) = n!\ a_n \quad \Rightarrow \quad a_n = \frac{P_n^{(n)}(0)}{n!}$$

$$\Rightarrow P_n(x) = P_n(0) + P_n'(0)x + \frac{P_n''(0)}{2!}x^2 + \frac{P_n'''(0)}{3!}x^3 + ... \frac{P_n^{(n)}(0)}{n!}x^n$$

Verbalise: The Maclaurin polynomial at x is the polynomial at 0 + 1st derivative at 0 times x to 0 + 2nd derivative at 0 times x squared over $2!$ + ….

17.3 The Taylor polynomials

In the same way we can prove that any polynomial can be re-written as

the Taylor polynomial: $P_n(x) = P_n(a) + P_n'(a)(x-a) + ... \frac{P_n^{(n)}(a)}{n!}(x-a)^n$

Verbalise: The Taylor polynomial at x is the polynomial at a + derivative at a times the signed distance from x to a + ….

Note: in the Maclaurin series x can be viewed as the signed distance from x to 0.

17.4 The Taylor series

Many differentiable functions $f(x)$ can be represented using the infinite

Taylor series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-2)^2 + ...$

Verbalise: The function value at x is the function value at a + derivative at a times the distance to a + ….

These functions can be **approximated** as polynomials using **truncated Taylor series** (that is, Taylor polynomials).

Consider the approximation error **Optional**

$$e_n(a, x) \equiv f(x) - P_n(a, x)$$

If the error $e_n(x) \to 0$ as $n \to \infty$, then $f(x)$ equals its Taylor series at x and $y(x) \approx P_n(a, x)$ is an approximation based on the n-*th* order Taylor polynomial.

For e^x, $\sin x$, $\cos x$, $e_n(x - a) \to 0$ faster than $(x - a)^n$

Proving the above results and checking the limiting behaviour of the error e_n is rather involved.

17.5 The Maclaurin series

If $a = 0$, the Taylor series is called the **Maclaurin series.**

Examples:

1. Find the Maclaurin series for e^x.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = e^x,$ $f'(x) = e^x,$ $f''(x) = e^x$ $\ldots$ $f^{(n)}(x) = e^x$	$f(0) = 1$ $f'(0) = 1$ $f''(0) = 1$ $\ldots$ $f^{(n)}(0) = 1$

Step 4. Substitute the above values into the general the Maclarin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \ \ldots$$

$\Rightarrow e^x \approx 1 + x, |x| < 1$ (in the vicinity of 0 the graph of e^x is almost a straight line)

2. Find the Maclaurin series for $\sin x$.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f'(x) = \cos x$	$f'(0)=1$
$f''(x) = -\sin x$	$f''(0)=0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(IV)}(x) = \sin x$	$f^{(IV)}(0)=0$
…	…

Step 4. Substitute the above values into the general Maclarin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$\Rightarrow \sin x \approx x, |x| < 1$ (in the vicinity of 0 the graph of $\sin x$ is almost a straight line)

3. Find the Maclaurin series for $\cos x$.

Solution:

Step 1. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = \cos x$	$f(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -1$

Step 4.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \quad -$$

$\Rightarrow \cos x \approx 1 - \frac{x^2}{2}, |x| < 1$ (in the vicinity of 0 the graph of $\cos x$ looks like a parabola – reflected **wrt** (with respect to) the horizontal axis shifted up by 1).

Thus, while any polynomial can be **re-written** as a Taylor or MacLaurin polynomial, many other n time differentiable functions can be **approximated** by Taylor or MacLaurin polynomials $P_n(x)$. The discussion on whether such approximation is possible or how to

decide the optimal degree of the approximating polynomial lies outside the scope of these notes.

17.6 L'Hospital's rule

Let us find $\lim_{x\to a}\frac{f(x)}{g(x)}$ when $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$, so that we have indeterminate form $0/0$.

In this case we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)(x-a)+\ldots}{g'(a)(x-a)+\ldots}\overset{:(x-a)}{=}\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

The recipe

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$$

is called **L'Hospital's** rule.

If still get the $0/0$ indeterminate form – continue applying the rule until arrive at the determinate answer (find the second derivatives of the numerator and denominator, their third derivatives *etc.*)

L'Hospital's rule can be extended to cover functions with just one derivative, $a=+\infty$ or $-\infty$ and ∞/∞ indeterminate forms but these proofs lie outside the scope of these notes.

Examples:

1) Find $\lim_{x\to 0}\frac{\sin x}{x}$.

Solution: $\lim_{x\to 0} x=\lim_{x\to 0}\sin x=0$. Applying L'Hospital's rule, $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$

2) Find $\lim_{x\to 0}\frac{1-e^x}{x}$.

Solution: $\lim_{x\to 0} x=\lim_{x\to 0} 1-e^x=0.$ Applying L'Hospital's rule, $\lim_{x\to 0}\frac{1-e^x}{x}=\lim_{x\to 0}\frac{-e^x}{1}=-1$

17.7 Applications

The above series are used by calculators to give approximate values of elementary functions, sin, cos, exponent *etc.*

17.8 Instructions for self-study

- **Revise ALGEBRA Summary (smile rule)**
- **Revise Lecture 5 (polynomials)**
- **Revise FUNCTIONS Summary**

- **Revise Lectures 13 - 14 (limits)**
- **Revise Lectures 14 - 15 (differentiation) and Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Revise Lecture 16 using the STUDY SKILLS Appendix**
- **Study Lecture 17 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find the first three non-zero terms in the Maclaurin series for $y = \frac{1}{1+x}$

b) Use the first two terms of the above Maclaurin series to approximate $y = \frac{1}{1+x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\frac{4}{5}$

d) Find $\lim_{x \to -1} [\frac{1}{1+x}(1 - e^{1+x})]$.

Q2.

a) Find the first three non-zero terms in the Maclaurin series for $y = \frac{1}{1-x}$

b) Use the first two terms of the above Maclaurin series to approximate $y = \frac{1}{1-x}$ in the vicinity of $x = 0$.

c) Find, without using a calculator, the approximate value of $\frac{4}{3}$

d) Find $\lim_{x \to 1} [\frac{1}{1-x} \sin(1-x)]$

Q3.

a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1+x}$

b) Use the first two terms of the above Maclaurin series to approximate $y = \sqrt{1+x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{3}}$

d) Find $\lim_{x \to -1} \frac{\sqrt{1+x}}{1-x^2}$

Q4.

a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1-x}$

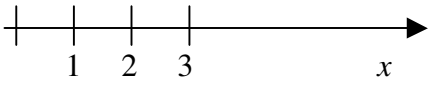
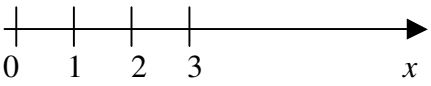
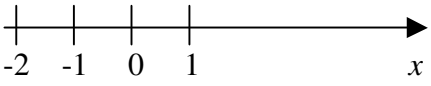
b) Use the first two terms of the above Maclaurin series to approximate $y = \sqrt{1-x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{5}}$

d) Find $\lim_{x \to 1} \frac{\sqrt{1-x}}{1-x^2}$

IV. SUMMARIES

Algebra Summary

OPERATIONS	TYPES OF VARIABLES
Addition (direct operation) Addition of whole numbers gives whole number 1. $a + b = b + a$ **Terminology:** a and b are called **terms** $a + b$ is called **sum** 2. $(a + b) + c = a + (b + c)$	**Whole numbers** are 1, 2, 3, ... 
Subtraction (inverse operation) **Def :** $a - b = x$: $x + b = a$ **Note:** $a + b - b = a$ **(subtraction undoes addition)** $a - b + b = a$ **(addition undoes subtraction)** 3. $a + 0 = a$ 4. for each a there exists one **additive inverse** $-a$: $a + (-a) = 0$ **Rules** (follow from Laws): $+ (b + c) = + b + c$ $+ a + b = a + b$ $- (- a) = a$ $- (a) = -a$	introduces **0** and **negative numbers**: $a - a = 0$ if $b > a$ $a - b = - (b - a)$ **Natural numbers** are 0, 1, 2, ...  **Integers** are ..., -2, -1, 0, 1, 2, ... 

Multiplication (direct operation)

For whole numbers n

$a \cdot n = \underbrace{a + \ldots\ldots\ldots\ldots + a}_{n \text{ terms}}$

Notation: $ab = a \cdot b = a \times b$

$2b = 2 \cdot b = 2 \times b$

$23 \neq 2 \cdot 3,\ 23 = 2 \cdot 10 + 3$

$2\frac{1}{2} \neq 2 \cdot \frac{1}{2},\ 2\frac{1}{2} = 2 + \frac{1}{2}$

$2\frac{3}{2} = 2 \cdot \frac{3}{2}$

1. $a \cdot b = a \cdot b$

Terminology: a and b are called **factors**

ab - **product**

2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Conventions: $abc = (ab)c$

$a(-bc) = -abc$

3. $a(b+c) = ab + ac$

$\longrightarrow$ Removing brackets

$\longleftarrow$ Factoring

4. $a \cdot 0 = 0$
5. $a \cdot 1 = a$

Rules (follow from Laws):

$(a + b)(c + d) = ac + ad + bc + bd$ (**SMILE RULE**)

$(-1) \cdot n = -n$

$(-1) \cdot (-1) = 1$

Division (inverse operation)

Def: $a/b = x$: $xb = a$

Terminology: a – **numerator**

b – **denominator**

a/b – **fraction (ratio)**

- **proper fraction** if $|a| < |b|$, a, b integers

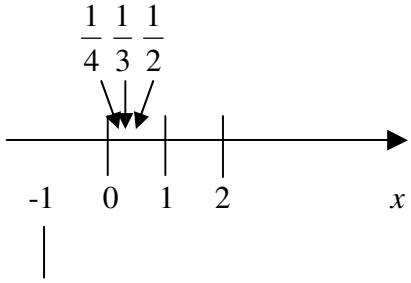
Note: $ab/b = a$ **(division undoes multiplication)**

$(a/b)b = a$ **(multiplication undoes division)**

6. For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $a \cdot 1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\frac{m}{n}$, where m and $n \neq 0$ are integers (**division by zero is not defined**)



Rules:

$$\frac{a}{b} \cdot n = \frac{an}{b}$$

$$\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$$

$$\frac{an}{bn} = \frac{a \cdot \cancel{n}}{b \cdot \cancel{n}} = \frac{a}{b}$$

CANCELLATION

$$\frac{\frac{1}{n}}{\frac{m}{n}}$$

FLIP RULE

$$\frac{a}{\cancel{b}} + \frac{c}{\cancel{b}} = \frac{a+c}{b}$$

Note: $\frac{a+c}{b} = (a+c)/b$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd}$$

COMMON DENOMINATOR RULE

n-th **power** b^n (direct operation)

If n – a whole number

$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}$

Rules

$a^m \cdot a^n = a^{m+n}$

(product of powers with the same base is a power with indices added)

$a^n \cdot b^n = (ab)^n$

(product of powers is power of product)

$a^m / a^n = a^{m-n}$

(ratio of powers is power of ratios)

$a^m / b^m = (a/b)^m$

(ratio of powers with the same base – subtract indices)

$a^0 = 1$

$a^{-n} = 1/a^n$

$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

n-th **root** (inverse to taking to power n)

Def: $\sqrt[n]{b} = x$: $x^n = b$

Note:

$\sqrt[n]{b^n} = b$

(taking *n-th* root undoes taking *n-th* power)

$(\sqrt[n]{b})^n = b$

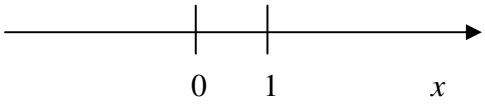
(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = \sqrt[n]{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$ …

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole number line (called **real line** for this reason)



Logarithm base b (inverse to taking b to power)

Def: $\log_b a = n$: $b^n = a$

Note:

$\log_b b^n = n$ (check using definition: $b^n = b^n$)

(taking $\log_b$ undoes taking b to power)

$b^{\log_b n} = n$

(taking b to power undoes $\log_b$)

introduces **irrational** (not rational) **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers**, $\sqrt{-1}, \log_{10}(-1)$, *etc.*

Rules (follow from Rules for Indeces):

$\log_b xy = \log_b x + \log_b y$
(log of a product is sum of logs)

$\log_b x/y = \log_b x - \log_b y$
(log of a ratio is difference of logs)

$\log_b 1 = 0$ **(log of 1 is 0)**

$\log_b b = 1$

$\log_b 1/a = -\log_b a$

$\log_b x^n = n \log_b x$
(log of a power is power times log)

$\log_b a = \log_c a/\log_c b$
(changing base)

General remarks

1. $a - b = a + (-b)$ ⟶ a difference can be re-written as a sum

2. $\frac{a}{b} = a \cdot \frac{1}{b} = ab^{-1}$ ⟶ a ratio can be re-written as a product

3. $\sqrt[n]{b} = b^{1/n}$ ⟶ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

6. In any algebraic expression operations should be performed using the Order of Operations convention.

Functions Summary

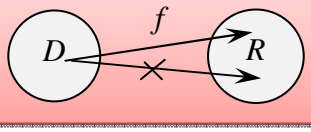
Variables are denoted mostly by $x, y, z, p, \ldots. w$.
A variable can take any value from an set of allowed numbers.

Functions are denoted mostly by f, g and h or $f(\ ), g(\ )$ and $h(\ )$
(no multiplication sign is intended!).

In mathematics the word *function* has three meanings:

1. $f(\ )$ - **an operation or a chain of operations** on an independent variable;
2. $f(x)$ - a **dependent variable**, that is the variable obtained when $f(\ )$ acts on an independent variable x.
3. Set of pairs $\{(x, f(x): f(\ )$ assigns one value $f(x)$ to every allowed value of $x\}$

A diagrammatical representation of a function



To specify a function we need to specify a **rule** - (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, $f(x)$ assigns one and only one value $y \in R$ (**range**, the set of all possible values of the dependent variable).

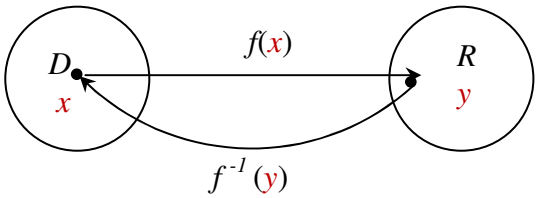
Inverse functions

$f^{-1}(x)$: $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function



Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. By convention, the **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**, and some authors do not bracket **arguments** of elementary **functions**, such as $\exp$, $\log$, $\sin$, $\cos$, $\tan$ *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.*

Other **operations** must be performed in the order of decreasing complexity, which is

oiB - **operations in Brackets**

F - Functions $f(\,)$

P- Powers (including inverse operations of roots and logs)

M - Multiplication (including inverse operation of division)

A - Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of operations (OOO)

1. Make implicit (invisible) brackets visible (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO

oiB	f( )	P	M	A
(including implicit)		$(\,)^{(\,)}$	×	+
		roots	÷	-
		logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2 + bx + c\,,$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared): b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2 + bx + c = 0\,.$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be $360°$ or 2π (radians)

$$\Rightarrow 2\pi\ (rad) = 360°$$
$$\Rightarrow 1\ (rad) \approx 57°$$

Above, x is the number of **radians** (angles $\approx 57°$) in a given angle (*cf*: $x\ (\mathrm{m}) = x \cdot 1\mathrm{m}$, so that x is the number of **units of length** (segments $1\mathrm{m}$ long) in a given segment)

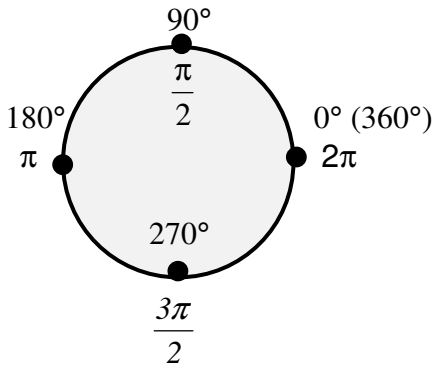
(**Note**: ***cf.*** means ***compare***)

The radian is a dimensionless unit of angle.

$$\Rightarrow\ x\ (rad) = x\frac{180^o}{\pi(\mathrm{rad})} = y^0,\ y^0 = y\frac{\pi(rad)}{180^0} = x\ (rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

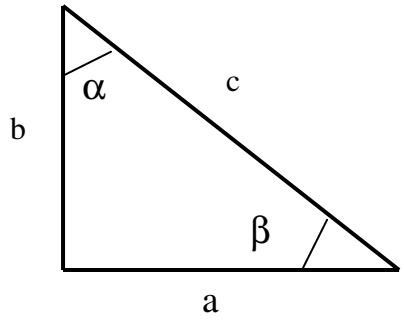
Frequently used angles



$30° = (\frac{30\pi}{180}=)\ \frac{\pi}{6}$	$45° = \frac{\pi}{4}$
$60° = (\frac{60\pi}{180}=)\ \frac{\pi}{3}$	
$120° = (\frac{120\pi}{180}=)\ \frac{2\pi}{3}$	

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90^0) as follows:



$$\sin\alpha = \cos\beta = \frac{a}{c}$$

$$\cos\alpha = \sin\beta = \frac{b}{c}$$

$$\tan a = \cot\beta = \frac{a}{b}$$

$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin\frac{\pi}{6} = \frac{1}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Trigonometric identities

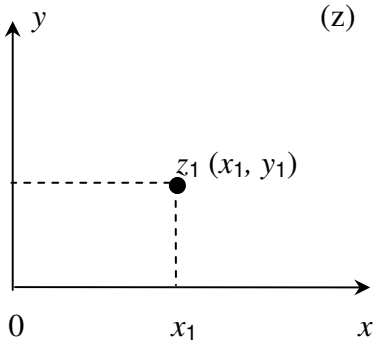
$\sin^2 x + \cos^2 x = 1$
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos 2x = \cos^2 x - \sin^2 x$
$\sin 2x = 2\sin x \cos x$

$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x + \alpha)$, where $\tan\alpha = \frac{A}{B}$

Complex Numbers

The Cartesian representation of a complex number

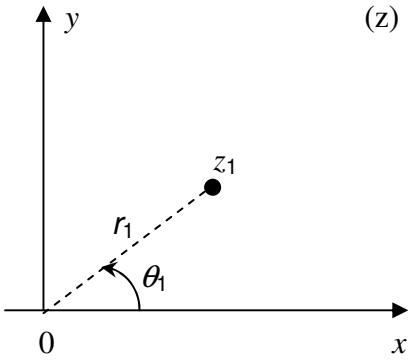


real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z = x + jy$, where x, y – real, $j = \sqrt{-1}$

x-coordinate y-coordinate (on the ordinary plane)

The exponential representation of a complex number



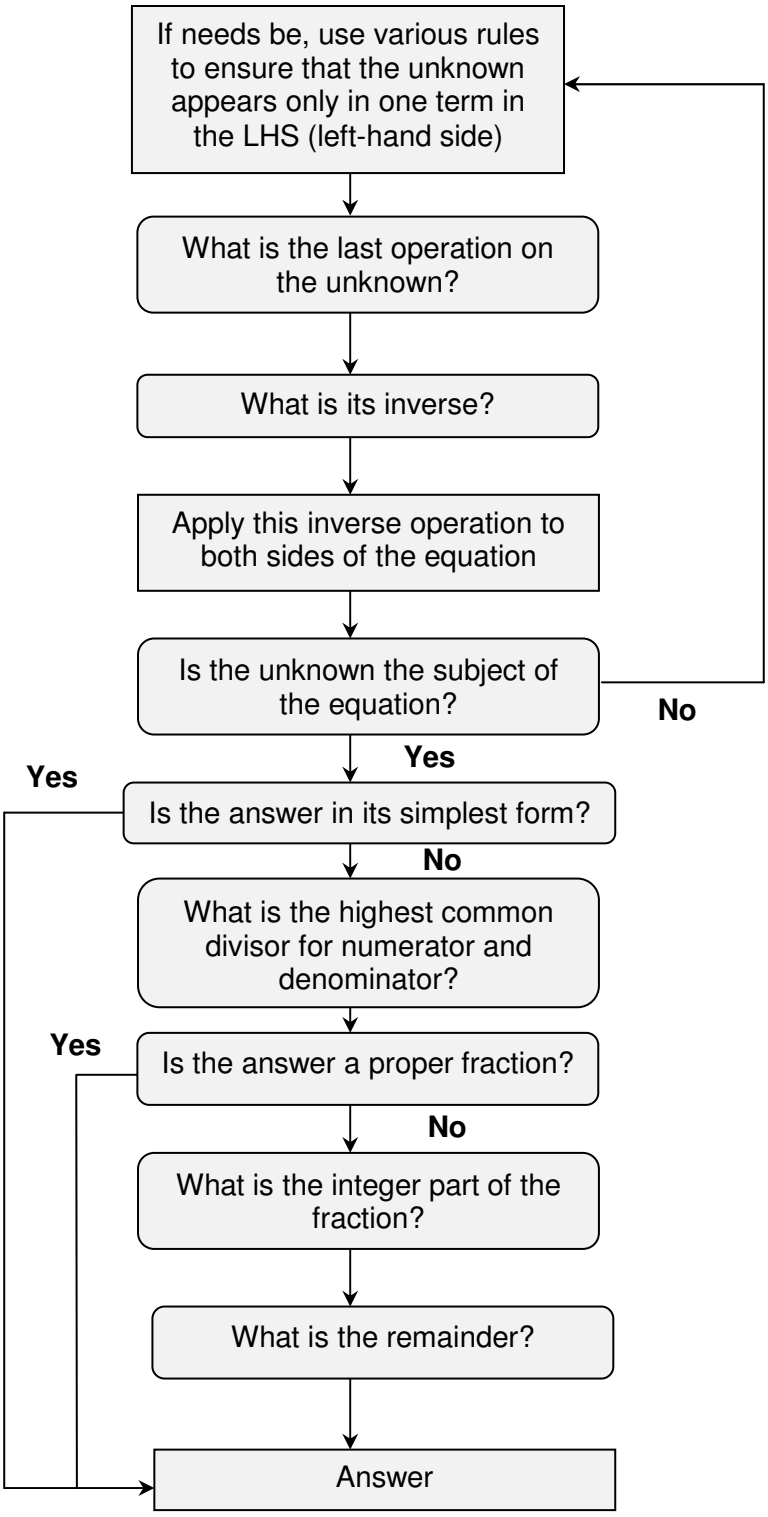
magnitude, $|z|$ **argument,** $\arg(z)$ (on the Argand diagram)

$z = r\, e^{j\theta}$, where $r \geq 0$, θ - angle (in degrees or radians)

polar radius r (distance to the origin)

polar angle θ (on the ordinary plane)

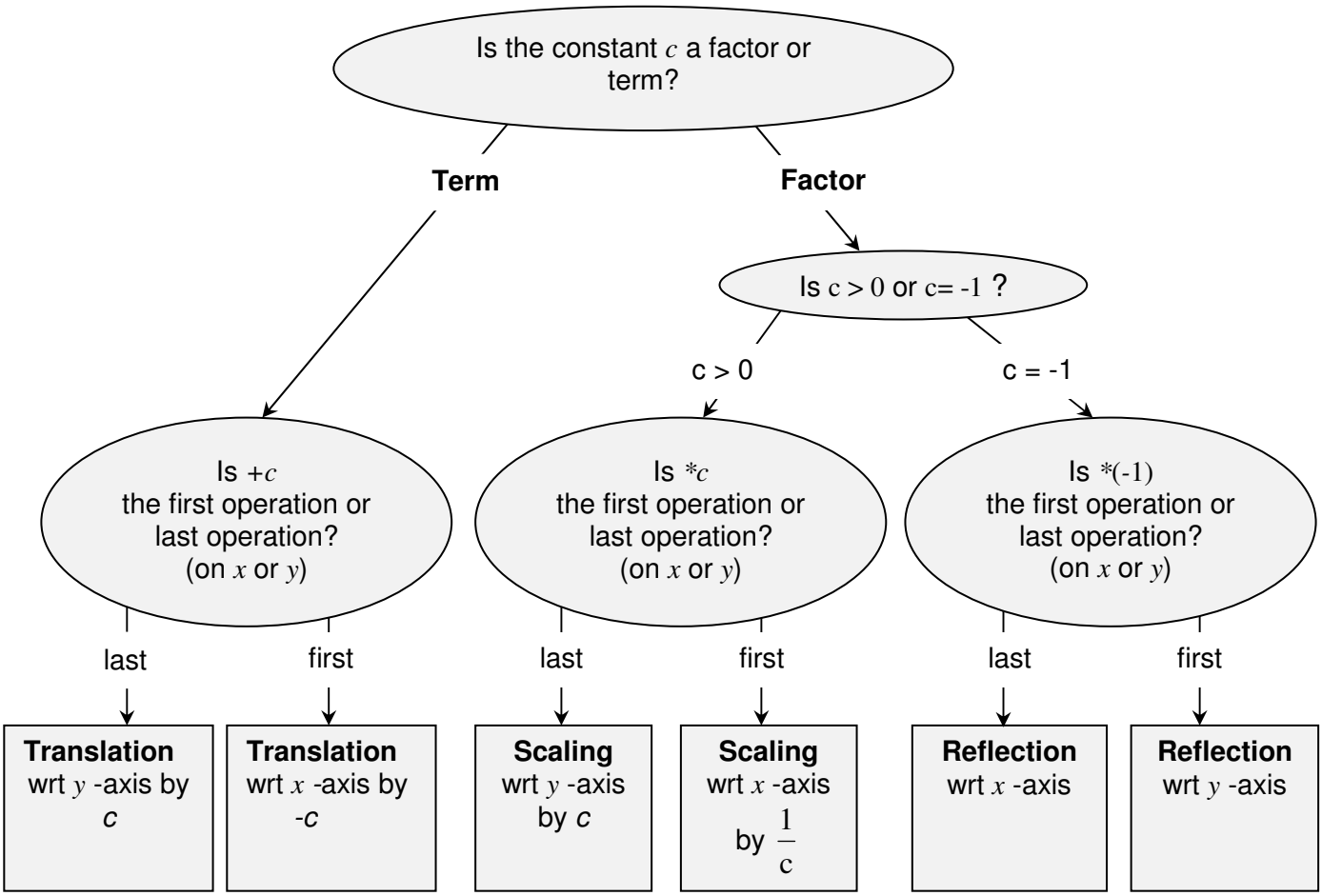
Decision Tree For Solving Simple Equations



Sketching Graphs by Simple Transformations

1. Drop all constant factors and terms.
2. Bring the constants back one by one in the Order of Operations (not necessary but advisable) and at each Step use the Decision Tree given below to decide which simple transformation is effected by each constant. Sketch the resulting graphs underneath one another.

DECISION TREE



Note 1: if c is a **negative factor**, write $c = (-1)^* |c|$, so that c effects two simple transformations and not one.
Note 2: if c is effects **neither first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations.
Note 3: if **the same operation** is applied to x in all positions in the equation this operation still should be treated as the first operation, as in $\ln(x - 2) + \sin(x - 2)$.
Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is effected by adding a constant c to y or multiplying it by a constant c, y should be made the subject of the functional equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\frac{1}{c}$, respectively.

Finding a Limit of a Sequence

To find a limit of a sequence (as $k \to \infty$) use the first principles (graphical representation) or

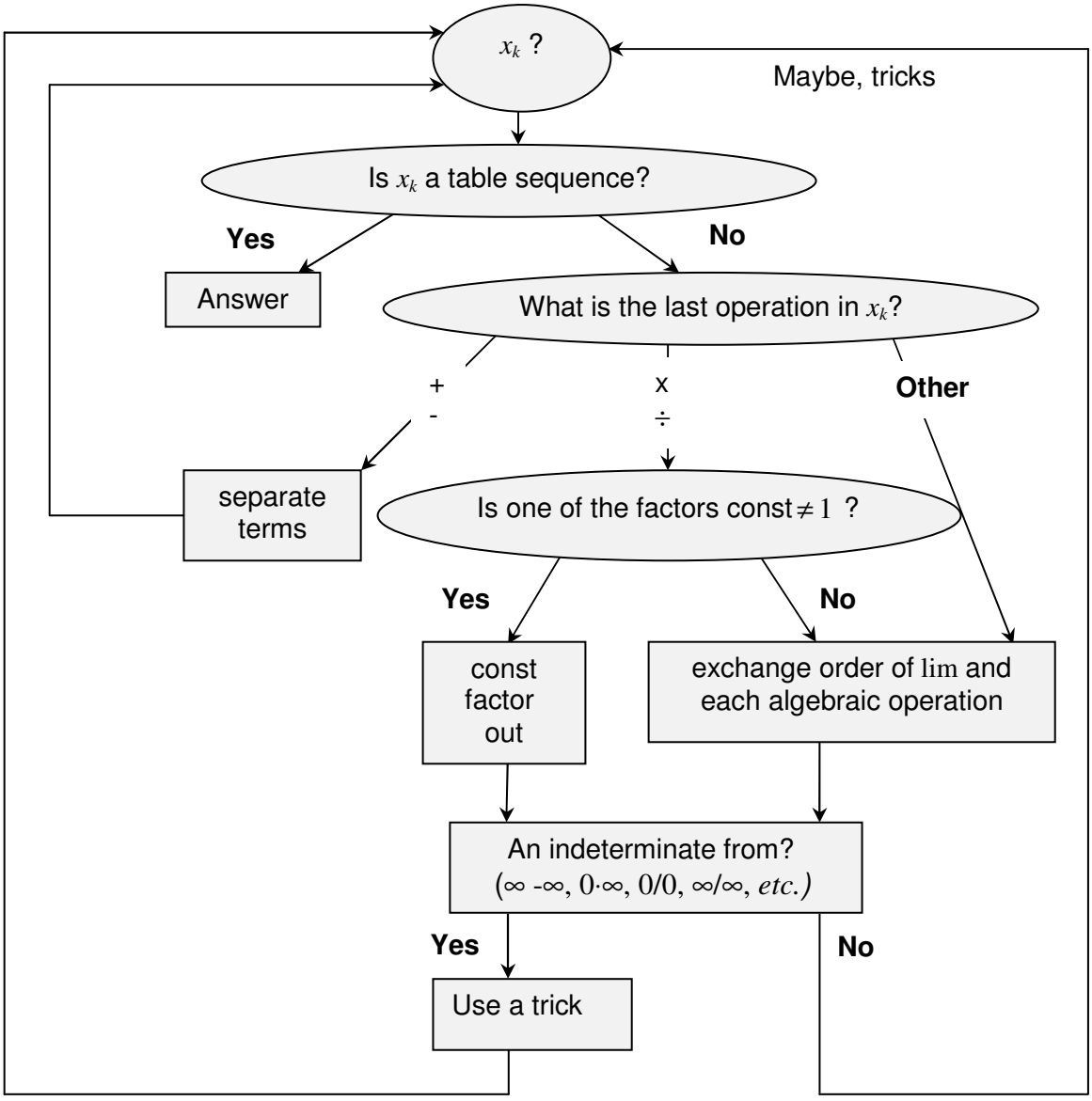
TABLE

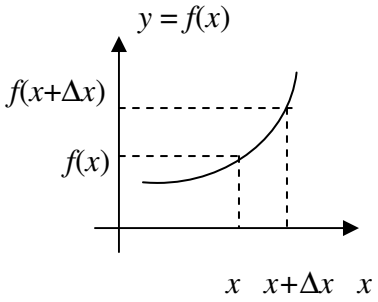
x_k	$\lim x_k$
constant	Constant
k	∞
$\frac{1}{k}$	0
$\frac{1}{a^n}$ $(a > 0)$	1
q^n $(\|q\| < 1)$	0

RULES

1. $\lim \alpha x_k = \alpha \lim x_k$ – *constant factor out*
2. $\lim [x_k \pm y_k] = \lim x_k \pm \lim y_k$ – *sum rule (separate terms)*
3. $\lim (x_k\, y_k) = \lim x_k \lim y_k$ – *product rule*
4. $\lim \frac{x_k}{y_k} = \frac{\lim x_k}{\lim y_k}$ – *quotient rule*
5. $\lim f(x_k) = f(\lim x_k)$
   limit and any algebraic operation or simple function can exchange places (commute)

DECISION TREE





Differentiation Summary

To differentiate a function use the first principles

Def: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

or represent roots and fractions as powers, make invisible brackets visible and use

TABLE

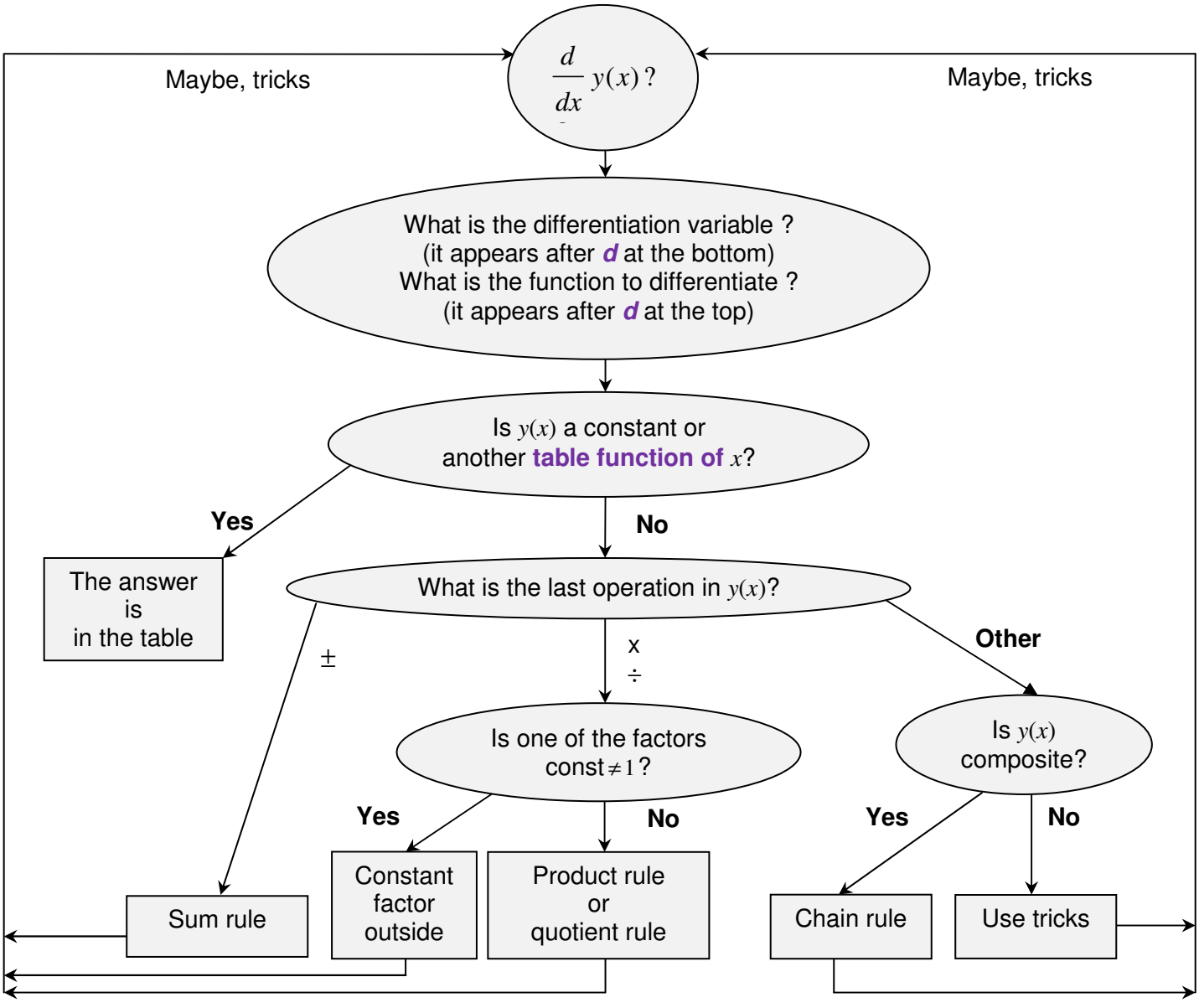
$f(x)$	$\frac{d}{dx} f(x)$
constant	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

RULES

1. $\frac{d}{dx}[\alpha f(x)] = \alpha \frac{df(x)}{dx}$ - *constant factor out*
2. $\frac{d}{dx}[f(x) + g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$ – *separate terms*
3. $\frac{d}{dx}[f(x)g(x)] = \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x)$ – *product rule*
4. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{df(x)}{dx} g(x) - \frac{dg(x)}{dx} f(x)}{g^2(x)}$ – *quotient rule*
5. $\frac{d}{dx}[f(g(x)] = \frac{df(g(x))}{dg(x)} \frac{dg(x)}{dx}$ – *chain rule*

decompose, differentiate, multiply)

DECISION TREE



V. GLOSSARY

ABSTRACTION - a general concept formed by extracting common features from specific examples.

ACUTE ANGLE – a positive angle which is smaller than 90^0.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by by a common divisor..

CIRCLE – a LOCUS of points inside and on a CIRCUMFERENCE.

CIRCUMFERENCE – a LOCUS of points at the same distance from a specific point (called a centre).

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

COMPOSITION – a combination of two or more functions (operations) forming a single new function by applying one to the output of another, e.g. composition of two function '$f(\,)$' and '$g(\,)$' is $f(g(\,))$.

CONCEPT - a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN, e.g. in the EXPRESSION $x + a$, x is usually understood to represent a VARIABLE and a, a constant (which is independent of this VARIABLE).

If there is only one algebraic CONSTANT, the preferred choice for its algebraic symbol is a. The second choice is b, and the third, c. If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM - a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s, kg/m^3,* £.

DOMAIN – a set of all allowed values of the ARGUMENT.

EQUATION – a mathematical statement involving UNKNOWNS and the = sign which can be true or false, e.g. $2x + 3y = 10$ is true when $x = y = 2$ but not when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. $2(a + b)$, $2ab$.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, *a* and *b*.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

FUNCTION

1. $f(\ )$ - an operation or a chain of operations on an DEPENDENT VARIABLE

2. $f(x)$ - a DEPENDENT VARIABLE, that is the variable obtained when $f(\ )$ acts on an independent variable x.

3. Set of pairs $\{(x, f(x)): f(\ )$ assigns one value $f(x)$ to every allowed value of $x\}$

To give an example,. '$f(x) = 2x + 3$, x - real' describes a FUNCTION whose SYMBOL is 'f' or '$f(\ )$' (no multiplication is implied) and whose ARGUMENT is called 'x'. Its DOMAIN is 'all reals'. Given any value of 'x', you can find the corresponding value of this FUNCTION by first multiplying the given value of 'x' by 2 and then by adding 3 to the result.

FUNCTION SYMBOL – usually a Latin letter, usually from the first part of the alphabet. If there is only one FUNCTION, the preferred choice for its symbol is 'f'. The second choice is 'g', and the third, 'h'. If there are more variables, then one chooses, in order of preference, 'u', 'v' and 'w'. If more FUNCTIONS are involved, we often make use of subscripts and superscripts, upper case and Greek letters.

GENERALISATION - an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH - a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INDEPENDENT VARIABLE – ARGUMENT, INPUT, e.g. given '$y = f(x)$', 'x' is an INDEPENDENT VARIABLE.

INPUT – (value of) ARGUMENT, (value of) INDEPENDENT VARIABLE, e.g. given '$y = f(x)$', 'x' is INPUT; also given '$y = f(2)$', '2' is INPUT.

INTEGER PART – when dividing a positive integer m into a positive integer n, k is the INTEGER PART if it is the largest positive integer producing $k*m \leq n$. The REMAINDER is the difference $n - k*m$, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that undoes what the original OPERATION does

LAST OPERATION – see ORDER OF OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION – an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. '$2x - 3 = 0$' is a LINEAR EQUATION, '2' is a COEFFICIENT in front of the UNKNOWN and '-3' is a FREE TERM.

LOCUS – a set of all points with a specific property.

NECESSARY - 'A' is a NECESSARY condition of 'B' IF 'B $\Rightarrow$ A' (B implies A), so that 'B' cannot take place unless 'A' is satisfied.

NON-DIMENSIONAL QUANTITY - a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. '$2 \ln(x) - 3 = 0$' is a NON-LINEAR EQUATION N in x.

OPERATION (mathematical) – action on CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which the OPERATIONS must be performed. By convention, the ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for are fractions and functions. Indeed, when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent, and some authors do not bracket ARGUMENTS of elementary FUNCTIONS, such as $\exp, \log, \sin, \cos, \tan$ *etc*. In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.* Other OPERATIONS must be performed in the order of decreasing complexity, which is

FUNCTIONS $f(\ )$

POWERS (including inverse operations of roots and logs)

MULTIPLICATION (including inverse operation of division)

ADDITION (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

OUTPUT – (value of) DEPENDENT VARIABLE, (value of the) FUNCTION, e.g. given '$y = f(x)$', 'y' is OUTPUT; also given '$f(x) = 2x + 3$' and '$x = 2$', '7' is OUTPUT (indeed, $2*2+3 = 7$).

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, e.g. 'ab' is a PRODUCT, and so is '$(a + b)c$'.

QUOTIENT - a mathematical expression where the last operation is division.

RANGE – the set of all possible values of the FUNCTION.

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the '=' sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SEQUENCE – a function with an INTEGER ARGUMENT.

SIMPLE EQUATION – an EQUATION that can be rearranged to contain the unknown in one term only.

SIMPLE TRANSFORMATIONS - translation, scaling or reflection.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT - 'A' is a SUFFICIENT condition of 'B' IF 'A $\Rightarrow$ B' (A implies B), so that if 'A' is satisfied, then 'B' takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is addition, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value or expression can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its algebraic symbol is x. The second choice is y and the third, z. If there are more variables, then one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

VI. STUDY SKILLS FOR MATHS

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours each week**:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.

2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**

3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).

4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

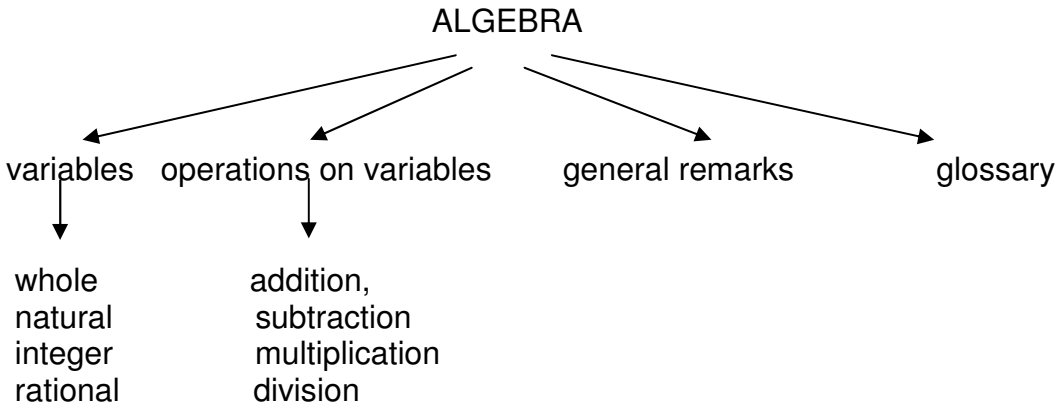
Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

Thus, you can construct the flow chart which looks like that



2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.

3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing.** Try this a few times before checking with the notes which step is a problem.

5. Repeat the process for each problem.

6. Repeat the process for each subtopic.

7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.

2. Keep consulting the **Glossary**.

3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.

4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to “make invisible brackets visible”.

2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.

3. The concept of **inverse operation**.

4. The **Decision Tree for Solving Simple Equations**.

5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.

6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real funciton of real variable (real argument). You should know the precise meaning of the word constant (you should always say "constant with respect to (the independent variable) x or t or whatever..."

7. How to do function composition and decomposition using **Order of Operations**

8. How to use graphs

9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos

10. The **Trigonometry Summary**

11. The approximate values of $e\ (\approx 2.71)$ and $\pi\ (\approx 3.14)$

12. What is $j\ (=\sqrt{-1}\ )$ and what is $j^2\ (=-1)$

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand digram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

15. The **Differentiation Summary**

16. The **Sketching by Simple Transformation Summary**

17. The **Limits Decision Tree** and basic indeterminacies

18. Sketching by analysis

VII. TEACHING METHODOLOGY (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question - Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft's book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGIG AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do an exercise yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) - rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.

Q: I also feel that the pace has been too high; we spent only about an hour on sequences and 2 weeks on integration.

A: We have only 22 weeks in one year to cover the necessary material, and it is spread out accordingly. However, you do not **need** to know more about sequences than was already discussed.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education - development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you - in its essence - for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise just a sequence of names, facts or dates there are well established techniques promoted in various books on memory. They suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with this objects. However, this technique will not work with technical information. What you need to establish are much deeper - meaningful - connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: The principles based on the first principles I feel are out of the question at our level in S1 and should be left to S2 as a minimum.

A: These proofs are above the A-level, but are definitely the 1st year level. Everyone should be able to start these proofs on the right note, namely write out the appropriate definitions and then SUBSTITUTE the appropriate functions. Only the students who reached the 1st class level are expected to finish these proofs using various tricks, but everyone should be able to follow the proofs as given by their teachers in Solutions to Exercises and learn how the tricks work. Also, remember, that in real life you will hardly ever differentiate or integrate using differentiation or integration techniques, but remembering the definitions (a derivative is a local slope or a local rate of change and integral is a signed area between a curve and a horizontal axis) might come very handy in your engineering life.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now - in their full glory - LATER, in year 2 and 3, not to-day.

Hope this helps!

VIII. SOLUTIONS TO EXERCISES FOR SELF-STUDY (WORKED EXAMPLES)

Solutions to Exercises in Lecture 13

Q1. Find the limits:

a) $\lim \dfrac{n^2+n+1}{2n+1}$

Solution

$\dfrac{\infty}{\infty}$ **- an Indeterminate form, try a trick**

$$\lim \frac{n^2+n+1}{2n+1} = \lim \frac{n^2+n+1^{\backslash \div n^2}}{2n+1} = \lim \frac{\frac{n^2}{n^2}+\frac{n}{n^2}+\frac{1}{n^2}}{\frac{2n}{n^2}+\frac{1}{n^2}} = \lim \frac{1+\frac{1}{n}+\frac{1}{n^2}}{\frac{2}{n}+\frac{1}{n^2}} = \frac{1+0+0}{0} = \infty$$

$\lim \dfrac{n^2+n+1}{2n^2+1}$

Solution

$\dfrac{\infty}{\infty}$ **- an Indeterminate form, try a trick**

$$\lim \frac{n^2+n+1}{2n^2+1} = \lim \frac{n^2+n+1^{\backslash \div n^2}}{2n^2+1} = \lim \frac{\frac{n^2}{n^2}+\frac{n}{n^2}+\frac{1}{n^2}}{\frac{2n^2}{n^2}+\frac{1}{n^2}} = \lim \frac{1+\frac{1}{n}+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1+0+0}{2} = \frac{1}{2}$$

b) $\lim \dfrac{n^2+n+1}{2n^3+1}$

Solution

$\dfrac{\infty}{\infty}$ **- an Indeterminate form, try a trick**

$$\lim \frac{n^2+n+1}{2n^3+1} = \lim \frac{n^2+n+1^{\backslash \div n^3}}{2n^3+1} = \lim \frac{\frac{n^2}{n^3}+\frac{n}{n^3}+\frac{1}{n^3}}{\frac{2n^3}{n^3}+\frac{1}{n^3}} = \lim \frac{\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n^3}} = \frac{0+0+0}{2} = 0$$

Comments

1. Many students apply the **CANCELLATION RULE** when dividing a sum. It applies only to quotients of products (because, cancellations are allowed only when an operation is immediately followed by its inverse). When dividing a sum we use the following algebraic rule from right to left:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$ (see **Algebra Summary** and Lecture 1).

2. Some students do not remember that when dealing with sequences, if the limit-sign is used without mentioning the independent variable (counter) explicitly, then it is implied that the limit is taken as $n \to \infty$, that is as n grows larger and larger without bounds.

3. Some keep loosing the limit-sign before they evaluate the limit. Some put symbol $\Rightarrow$ instead of =. Some put the = sign after the limit-sign (which creates a meaningless sequence of symbols).

3. Please remember:

Division by zero is undefined, there is no number = $\frac{1}{0}$ (some students think it is zero). However, if as n grows, x_n grows without bounds this situation is described symbolically as

$$\lim x_n = \infty$$

For example, if a constant is divided by a smaller and smaller number the result grows larger and larger. Symbolically, the limit is said to be $\frac{1}{0}=\infty$ (we are talking limits here and not numbers).

Q2. Find the sum of a hundred terms of the arithmetic series with first term 1 and common difference 2.

Solution

$$S_N = N\frac{x_1 + x_N}{2}$$

In this problem x_N is not given, only x_1 and common difference, a, so x_N has to be found using these data and the recurrence formula that defines the arithmetic sequence, $x_{n+1} = x_n + a$.

So we can write

$$x_2 = x_1 + a$$
$$x_3 = x_2 + a = (x_1 + a) + a = x_1 + 2a$$
$$x_4 = x_3 + a = (x_1 + 2a) + a = x_1 + 3a$$
$$\ldots\ldots$$

Analysing the pattern, we get

$$x_N = x_1 + (N-1)a$$

It follows that

$$S_N = N\frac{x_1 + x_N}{2} = N\frac{x_1 + x_1 + (N-1)a}{2} = N\frac{2x_1 + (N-1)a}{2} = 100\times\frac{2 + (100-1)\times 2}{2} =$$
$$100\times\frac{2 + 99\times 2}{2} = 100\times\frac{2 + 198}{2} = 100\times\frac{200}{2} = 100\times 100 = 10{,}000$$

Q3. A steel ball-bearing drops on to a smooth hard surface from a height h. The time to first impact is $T = \sqrt{\frac{2h}{g}}$, where g is the acceleration due to gravity. The times between successive bounces are $2eT$, $2e^2T$, $2e^3T$,..., where e is the coefficient of restitution between the ball and the surface $(0 < e < 1)$. Find the total time taken up to the fifth bounce if $T = 1$ and $e = 0.1$.

Solution

The total time $t = T + 2eT + 2e^2T + 2e^3T + 2e^4T$.
Starting with the second term we have the geometrical sequence, with

$x_1 = 2eT$ and the common ratio $a = \frac{2e^2T}{2eT} = e$.

Therefore, $t = T + S_4$, where $S_N = \frac{x_1(1-a^N)}{1-a} = \frac{2eT(1-e^N)}{1-e}$.

Substituting $T = 1$, $e = 0.1$ and $N = 4$, we have

$$S_4 = \frac{2\times 0.1(1-0.1^4)}{1-0.1} = \frac{2\times 0.1(1-0.0001)}{1-0.1} = \frac{0.2\times 0.9999}{0.9} \approx 0.22$$
$$\Rightarrow t \approx 1 + 0.22 = 1.22.$$

Comments

Some students do not read the question carefully, use the formula for T, which is irrelevant here and some cannot come up with the correct formula for the total time to fifth bounce. **It is important to read every question carefully, several times if necessary, to understand what is being asked.**

Some students use S_∞ to calculate S_4 !

Q4 **(advanced).** The irrational number π can be defined as the area of a unit circle (circle of radius 1). It can be defined as a limit of a sequence of known numbers a_n. The method used by Archimedes was to inscribe in the unit circle a sequence of regular polygons (that is, polygons with equal sides – see figure 1 below). As the number of sides increases the polygon "fills" the circle. For any given number n find the area a_n of the regular polygon inscribed in the unit circle.

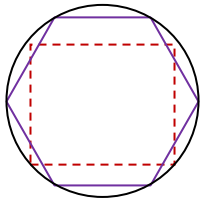


Figure 1. A circle with two inscribed regular polygons, one with four sides (a square) and one, with six.

a) Show, using of trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras theorem) and trigonometric identity $\cos 2\theta = 1 - 2\sin^2\theta$ that the area a_n satisfies the recurrence relation

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1 - \sqrt{\left[1 - \left(\frac{2a_n}{n}\right)^2\right]}, n \geq 4.$$

Solution

a) **For any number n the area of the regular polygon (see figure 2) can be written as**

$$a_n = na_\Delta,$$

where the area of each of n equilateral triangles is

$$a_\Delta = \sin\theta_n \cos\theta_n, \quad \text{where } \theta_n = \frac{360^0}{2n} = \frac{180^0}{n}.$$

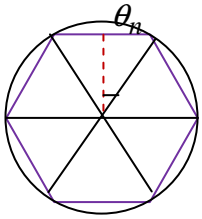


Figure 2. Calculation of the area of a hexagon inscribed in the unit circle.

Thus, we can reformulate the problem as follows: show that

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1-\sqrt{\left[1-\left(\frac{2a_n}{n}\right)^2\right]}, n \geq 4 \qquad (1)$$

where $a_n = n\sin\theta_n \cos\theta_n$

Step 1. Consider the LHS (left-hand side) of Eq. (1). It contains a_{2n}**. Since the sequence element** $a_n = n\sin\theta_n \cos\theta_n$ **is a function of** n**, we can write**

$$a_{2n} = 2n\sin\theta_{2n}\cos\theta_{2n}$$

Using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$ **we can write**

$$a_{2n} = n\sin 2\theta_{2n}\text{, where } \theta_{2n} = \frac{180^0}{2n} = \frac{\theta_n}{2}$$

This means that

$$a_{2n} = n\sin\theta_n$$

and LHS (left-hand side) of Eq. (1): $2\left(\frac{a_{2n}}{n}\right)^2 = 2\left(\frac{n\sin\theta_n}{n}\right)^2 = 2\sin^2\theta_n$

Step 2. Consider the RHS (right-hand side) of Eq. (1).

$$1-\sqrt{\left[1-\left(\frac{2a_n}{n}\right)^2\right]} = 1-\sqrt{\left[1-\left(\frac{2n\sin\theta_n\cos\theta_n}{n}\right)^2\right]} = 1-\sqrt{\left[1-(2\sin\theta_n\cos\theta_n)^2\right]} =$$

Using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$ **we can write**

$$1-\sqrt{\left[1-(\sin 2\theta_n)^2\right]} = 1-\sqrt{\left[1-\sin^2 2\theta_n\right]} =$$

Using Pythagoras's Theorem $\sin^2\theta + \cos^2\theta = 1$ **we can write**

$$\sin^2 2\theta_n + \cos^2 2\theta_n = 1 - \sin^2 2\theta_n$$

$$\Rightarrow \cos^2 2\theta_n = 1-\sin^2 2\theta_n$$

$$1-\sqrt{\cos^2 2\theta_n} = 1-\cos 2\theta_n$$

Note: although $\sqrt{\cos^2 2\theta_n}$ **could be** $\pm\cos 2\theta_n$**, when** $n \geq 4$ **all angles** $2\theta_n$ **are acute and therefore, all** $\cos 2\theta_n > 0$**.**

STEP 3. Using the trigonometric identity

$$\cos 2\theta = 1-2\sin^2\theta \;/ -\cos 2\theta, \; +2\sin^2\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

Therefore, LHS=RHS and we have shown that relationship (1) is correct.

b) Use trigonometry to show that $a_4 = 2$.

Solution

The area of the square is $a_4 = 4a_\Delta = 4\sin 45^0 \cos 45^0 = 4\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2} = 4\times\frac{2}{4} = 2.$

c) Use a) and b) to find a_{64}.

Solution

Using the recurrence relation in Eq. (1) we have

$$2\left(\frac{a_8}{4}\right)^2 = 1-\sqrt{\left[1-\left(\frac{2a_4}{4}\right)^2\right]} = 1-\sqrt{\left[1-\left(\frac{2\times 2}{4}\right)^2\right]} = 1$$

$$\Rightarrow \left(\frac{a_8}{4}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \frac{a_8}{4} = \sqrt{\frac{1}{2}}$$

$$\Rightarrow a_8 = 4\times\sqrt{\frac{1}{2}} = 2\sqrt{2} \approx 2.8$$

To continue, we have

$$2\left(\frac{a_{16}}{8}\right)^2 = 1-\sqrt{\left[1-\left(\frac{2a_8}{8}\right)^2\right]} = 1-\sqrt{\left[1-\left(\frac{2\times 2\sqrt{2}}{8}\right)^2\right]} = 1-\sqrt{\left[1-\left(\frac{\sqrt{2}}{2}\right)^2\right]} = 1-\frac{1}{\sqrt{2}}$$

$$\Rightarrow \left(\frac{a_{16}}{8}\right)^2 = \frac{1}{2}-\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\Rightarrow \frac{a_{16}}{8} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\Rightarrow a_{16} = 8\times\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \approx \sqrt{64\times\frac{\sqrt{2}-1}{2\sqrt{2}}} = \sqrt{32\times\frac{\sqrt{2}-1}{\sqrt{2}}} = \sqrt{16\times\sqrt{2}\times(\sqrt{2}-1)} = \sqrt{16\times(2-\sqrt{2})} \approx$$

$$4\times\sqrt{2-1.4} = 4\times\sqrt{0.6} \approx 4\times 0.77 = 3.08$$

That is, the larger the index n the closer the sequence element to 3.14 (the area of the unit circle is π). There are many more sequences whose limit is π. Some of them converge to π much faster than the sequence discussed here.

Q5. Explain why $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$.

Solution

$\sum_{k=1}^{\infty} x[k]$ **is the shorthand which uses the** $\sum$ **- notation to denote the series**

$x[1] + x[2] + x[3] + \ldots$

$\sum_{n=1}^{\infty} x[n]$ **is the shorthand which uses the** $\sum$ **- notation to denote the series**

$x[1] + x[2] + x[3] + \ldots$

Therefore, $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n]$**, that is the actual symbol used to denote the counter does not matter (we say "it gets summed out".)**

$\sum_{k=0}^{\infty} x[k+1]$ **is the shorthand which uses the** $\sum$ **- notation to denote the series**

$x[1] + x[2] + x[3] + \ldots$

Therefore, $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$**.**

Comments

1. When given a word problem, it is important to learn to reformulate is using mathematical langusge. Read the text of the problem several times. Look for the key words, such as "show", "prove", "simplify" *etc*. Make several attempts at condensing the text to understand what is required.

2. Most mathematical problems that ask you to "show" something require comparing the left hand side with the right hand side of a given relationship.

3. Q4 is a challenging question and the correct tactics would be to leave it till the end. However Q4 b) and Qc) are easy and it would pay off to try to do them first. This is an important examination technique and engineering skill too: **However difficult the problem, try to do as much as you can!**

4. While not everybody can solve Q4 themselves everybody should be able to follow the solution, because everybody should be able to use the trigonometric identities.

5. **The problem also gives students an idea what they should be able to do to achieve the first class result**.

Comments

Learn to understand the $\sum$ - notation – it is very common in engineering writings. Learn to read the Lecture Notes: the $\sum$ - notation was given there.

Do not necessarily assume that if you cannot solve the problem it is too difficult. The chances are, you just do not understand the mathematical language. Go back to the relevant Lecture or Summary and learn the meaning of the relevant words and symbols.

ALWAYS STUDY THE LECTURE FIRST – JUST AS THE STUDY SKILLS SUMMARY SUGGESTS.

ATTEMPT LECTURE EXERCISES ONLY AFTER THE LECTURE HAS BEEN UNDERSTOOD.

Solutions to Exercises in Lecture 14

Q1. Find

a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (using the table entry $\lim_{x\to 0}\frac{\sin x}{x}=1$ and trigonometric identities)

Solution

Substituting the limiting value of x (which is 0), that is, evaluating the function $\frac{1-\cos x}{x^2}$ at 0, we obtain the $\frac{0}{0}$ indeterminate form (as we always do when evaluating $\frac{dy}{dx}$ directly):

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1-\cos 0}{0^2}=\frac{1-1}{0}=\frac{0}{0}$$

Therefore to evaluate this limit we need tricks. Since the hint suggests the use of the table result,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

we have to keep modifying the function until we reduce it to a composition (chain) of simple operations on $\frac{\sin x}{x}$.

Step 1: Our function $\frac{1-\cos x}{x^2}$ and $\frac{\sin x}{x}$ each is a ratio of a trigonometric function and a power.

The first difference between them that we note is that the numerator of our function is $1-\cos x$ and the numerator of the table function is $\sin x$. However, $1-\cos(\,)$ can be expressed in terms of $\sin(\,)$ using the trigonometric identities,

$$\cos 2A=\cos^2 A-\sin^2 A=(1-\sin^2 A)-\sin^2 A=1-2\sin^2 A$$

(see the trigonometric identities in the Trigonometry Summary). Denoting the argument on the left by x, we have

$$x=2A$$

Using the Decision Tree for Solving Equations wit Only One Term Containing the Unknown we can solve this equation for A,

$$A=\frac{x}{2}$$

Substituting this into the above trigonometric identity, we have

$$\cos(x)=1-2\sin^2\frac{x}{2}$$

We can find $1-\cos(x)$, first by multiplying both sides by -1 to obtain

$$-\cos(x)=-1+2\sin^2\frac{x}{2}$$

and then by adding 1 to both sides. This gives us

$$1-\cos(x)=2\sin^2\frac{x}{2}$$

Thus we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

Step 2: Substituting the limiting value of x (which is 0), that is, evaluating $\frac{2\sin^2\frac{x}{2}}{x^2}$ at 0, we still obtain the $\frac{0}{0}$ indeterminate form. Therefore to evaluate the limit we need further tricks. Again, we note that function $\frac{2\sin^2\frac{x}{2}}{x^2}$ and $\frac{\sin x}{x}$ each is a ratio of a trigonometric function and power. One easy step to make them more alike is to get rid of the constant factor using the "constant factor out" rule for limits (see the Limits Summary):

$$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=2\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}$$

The main difference between $\frac{\sin^2\frac{x}{2}}{x^2}$ and $\frac{\sin x}{x}$ is the fact that our function contains powers. Using the definition of a power and the product rule for limits (see the Limits Summary) we can write

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=2\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}=$$

$$2\lim_{x\to 0}\left[\frac{\sin\frac{x}{2}}{x}\cdot\frac{\sin\frac{x}{2}}{x}\right]=2\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}=2\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\right)^2$$

Step 3. Our new function $\frac{\sin\frac{x}{2}}{x}$ is very similar to $\frac{\sin x}{x}$. The only difference is the fact that in our function the denominator is not the same as the argument of *sin*. We can make it the same by dividing the denominator x by 2, but to keep the whole function the same, we have to multiply this denominator by 2 as well, or else multiply the whole function by $1/2$:

$$\frac{\sin\frac{x}{2}}{x}=\frac{1}{2}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

Then

$$\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}=\lim_{x\to 0}\frac{1}{2}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

Using the "constant factor out" rule for limits again (see the Limits Summary) we have

$$\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}=\frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

and therefore,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=2\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\right)^2=2\left(\frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

The limit in brackets is almost the same as the table result we want to use. The only difference is that the independent variable that tends to 0 is not the same as the argument of $\sin$ and denominator. However, as $x\to 0$, $\frac{x}{2}\to 0$ and as $\frac{x}{2}\to 0$, $x\to 0$.

Therefore, these are equivalent statements and we can write

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=2\left(\frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=2\left(\frac{1}{2}\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

Step 4: Applying the table entry, $\lim_{x\to 0}\frac{\sin x}{x}=1$, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=2\left(\frac{1}{2}\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=2\cdot\left(\frac{1}{2}\cdot 1\right)^2$$

Using the Algebra rules "the factor 1 can always be dropped", the Order Of Operations and CANCELLATION RULE),

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=2\cdot\left(\frac{1}{2}\cdot 1\right)^2=2\cdot\left(\frac{1}{2}\right)^2=2\cdot\frac{1}{4}=\frac{1}{2}$$

b) $\lim_{x\to 0}\frac{e^x-e^{-x}}{2}$

Solution

Substituting the limiting value 0 for *x* we obtain a finite number:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2}=\frac{1-1}{2}=\frac{0}{2}=0$$

This is the answer.

c) $\lim_{x\to 0}\frac{\sin^2 3x}{x^3}$

Solution

Substituting the limiting value 0 for x we obtain the $\frac{0}{0}$ indeterminate form:

$$\lim_{x\to 0}\frac{\sin^2 3x}{x^3}=\frac{\sin^2 3\times 0}{0^3}=\frac{0}{0}$$

Therefore to evaluate this limit we need to use tricks. Modifying the function and using rules for limits,

$$\lim_{x\to 0}\frac{\sin^2 3x}{x^3}=\lim_{x\to 0}\frac{1}{x}\frac{\sin^2 3x}{x^2}=\lim_{x\to 0}\frac{1}{x}\left(\frac{\sin 3x}{x}\right)^2=\lim_{x\to 0}\frac{1}{x}\times\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)^2=$$

$$\lim_{x\to 0}\frac{1}{x}\times\left(\lim_{x\to 0}\frac{\sin 3x}{x}\right)^2=\lim_{x\to 0}\frac{1}{x}\times\left(3\lim_{3x\to 0}\frac{\sin 3x}{3x}\right)^2=\infty\cdot 9,$$ **since** $\lim_{x\to 0}\frac{\sin x}{x}=1.$

Thus, $\lim_{x\to 0}\frac{\sin^2 3x}{x^3}=\infty.$

d) $\lim_{x\to 0}\frac{x^2-1}{2x^2+3}$

Solution

Substituting the limiting value 0 for x we obtain a finite number:

$$\lim_{x\to 0}\frac{x^2-1}{2x^2+3}=\frac{0^2-1}{2\times 0^2+3}=-\frac{1}{3}$$

This is the answer.

e) $\lim_{x\to\infty}\frac{x^2-1}{2x^2+3}$

Solution

Substituting ∞ for x (doing this formally, remember ∞ is not a number!) we obtain the $\frac{\infty}{\infty}$ indeterminate form

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+3}=\frac{\infty^2-1}{2\times\infty^2+3}=\frac{\infty}{\infty}.$$

Therefore to evaluate the limit we need to use tricks:

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+3}\ \backslash\div x^2 = \lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2+\frac{3}{x^2}} = \frac{1-\frac{1}{\infty^2}}{2+\frac{3}{\infty^2}} = \frac{1}{2}.$$

Comments

1. Many students use the symbol of $\lim$ without due care. Some lose it half way through. This is the sign of an operation. You should drop it only after the operation has been carried out.

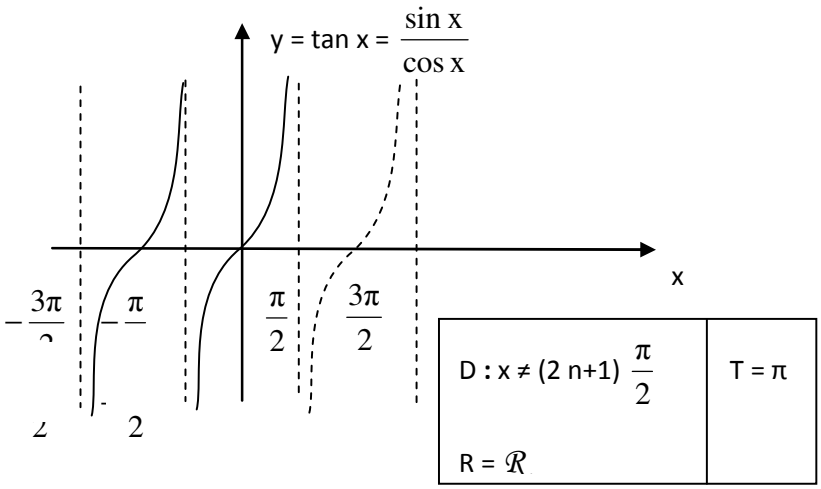
2. Some students try to use L'Hospital's rule to solve this problem. However, we have not studied L'Hospital's rule yet. **If a different method is used in tests and exams to the one specified in the problem statement most marks are lost even if the correct answer is achieved.**

3. Some students try to use the techniques appropriate for problem e) while attempting problem d). Division by the highest power is appropriate only when the independent (control) variable tends to ∞, because it gives rise to terms that in the limit are $\frac{1}{\infty}=0$. Of course, ∞ is not a number. The meaning of the formal statement $\frac{1}{\infty}=0$ is **"when dividing 1 by larger and larger numbers, in the limit we obtain 0."**

Q2. Discuss continuity of $y = \tan x$.

Solution

The graph of the function $y = \tan x$ has been discussed before.



The definition of continuity has been given in Lecture 14. A function is continuous on an interval if it is continuous at each point of the interval. A function is continuous at a point if the limit of the function at this point is the same as its value at this point. The function $y = \tan x$ has this property for all points in each natural domain $x \neq (2n+1)$, n – integer.

Q3. Find the rate of change (derivative) of function $y(t) = t^2 - 2t$ at $t = 1$.

Solution

The only way to find a derivative that has been discussed in Lecture 14 is by applying the basic principles (definition of the derivative).

Therefore we proceed by using this definition:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - 2(t+\Delta t) - (t^2 - 2t)}{\Delta t} =$$

$$y(t) = t^2 - 2t$$
$$y(t+\Delta t) = (t+\Delta t)^2 - 2(t+\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + (\Delta t)^2 - 2(t+\Delta t) - (t^2 - 2t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + (\Delta t)^2 - 2t - 2\Delta t - t^2 + 2t}{\Delta t} =$$

$$\lim_{\Delta t \to 0} \frac{2t\Delta t + (\Delta t)^2 - 2\Delta t}{\Delta t} = \lim_{\Delta t \to 0} (2t + \Delta t - 2) = 2t - 2 .$$

At $t = 1$, $\frac{dy}{dt}(1) = 2 - 2 = 0.$

Comments

When doing this exercise, most students use their knowledge of simple derivatives to arrive at the answer rather than the basic definition. Yet, you are supposed to practice the techniques explained in the Lectures. **Also, if a different method is used in tests and exams to the one specified in the problem statement most marks are lost - even if the answer is correct.**

Q4. Use the basic definition of a derivative to prove that

a) $\frac{d}{dx} x^n = n x^{n-1}$ (Hint: use the Binomial Theorem)

Solution

The only way to find a derivative that has been discussed in Lecture 14 is by applying the basic principles (definition). Therefore,

$$f(x) = x^n$$
$$f(x+\Delta x) = (x+\Delta x)^n$$

$$\frac{df(x)}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}=$$

$$\lim_{\Delta x\to 0}\frac{x^n+n(\Delta x)x^{n-1}+\frac{n(n-1)}{2!}(\Delta x)^2x^{n-2}+\ldots+(\Delta x)^n-x^n}{\Delta x}=$$

$$\lim_{\Delta x\to 0}\frac{n\Delta x\cdot x^{n-1}+\frac{n(n-1)}{2!}(\Delta x)^2\cdot x^{n-2}+\ldots(\Delta x)^n}{\Delta x}=$$

$$\lim_{\Delta x\to 0}(nx^{n-1}+\frac{n(n-1)}{2!}\Delta x\cdot x^{n-2}+\ldots\Delta x^{n-1})=nx^{n-1}$$

b) $\frac{d}{dx}e^x=e^x$

Solution

$$\frac{df(x)}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x}=$$

$$f(x)=e^x$$
$$f(x+\Delta x)=e^{x+\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{e^xe^{x+\Delta x}-e^x}{\Delta x}=\lim_{\Delta x\to 0}e^x\frac{e^{\Delta x}-1}{\Delta x}=\lim_{\Delta x\to 0}e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=\lim_{\Delta x\to 0}e^x=e^x,$$

where we used the table result $\lim_{x\to 0}\frac{e^x-1}{x}=1.$

c) $\frac{d}{dx}\sin x=\cos x$ **(advanced)**

Solution

$$\frac{df(x)}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}=$$

$$f(x)=\sin x$$
$$f(x+\Delta x)=\sin(x+\Delta x)$$

$$\lim_{\Delta x\to 0}\frac{\sin(x)\cos(\Delta x)+\cos(x)\sin(\Delta x)-\sin(x)}{\Delta x}=\lim_{\Delta x\to 0}[\frac{\sin(x)\cos(\Delta x)-\sin(x)}{\Delta x}+\frac{\cos(x)\sin(\Delta x)}{\Delta x}]=$$

$$\lim_{\Delta x\to 0}\frac{\sin(x)\cos(\Delta x)-\sin(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\cos(x)\sin(\Delta x)}{\Delta x}=$$

$$\sin(x)\lim_{\Delta x\to 0}\frac{\cos(\Delta x)-1}{\Delta x}+\cos(x)\lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x}=-\sin(x)\lim_{\Delta x\to 0}\frac{1-\cos(\Delta x)}{\Delta x}+\cos(x)=\cos(x)$$

since

$$\lim_{\Delta x\to 0}\frac{1-\cos(\Delta x)}{\Delta x}=$$

$$\lim_{\Delta x\to 0}\frac{2\sin^2(\frac{\Delta x}{2})}{\Delta x}=2\lim_{\Delta x\to 0}\sin(\frac{\Delta x}{2})\lim_{\Delta x\to 0}\frac{\sin(\frac{\Delta x}{2})}{\Delta x}=2\lim_{\Delta x\to 0}\sin(\frac{\Delta x}{2})\lim_{\Delta x\to 0}\frac{1}{2}\frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}=$$

$$2\times\frac{1}{2}\lim_{\Delta x\to 0}\sin(\frac{\Delta x}{2})\lim_{\frac{\Delta x}{2}\to 0}\frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}=\lim_{\Delta x\to 0}\sin(\frac{\Delta x}{2})\times 1=\sin\left(\frac{0}{2}\right)=\sin(0)=0$$

Comments

Few students attempt to arrive at the answer using the basic definition. Remember, you are supposed to practice the techniques explained in the Lectures. Also, **if a different method is used in tests and exams to the one specified in the problem statement most marks are lost - even if the answer is correct.**

This last problem is a bit of a challenge (the others were not – everyone could do them – if they studied the Lectures, rather than learning by rote). However, everybody should be able to follow the solution, because everybody should know the properties of limits (see Lectures and the **Limits Summary**) and know the basic definition of derivative. All table results used have been given in the Lecture Notes.

The problem gives students an idea what they should be able to do to achieve the first class result.

Solutions to Exercises in Lecture 15

Q1. Find

a) $\dfrac{d}{d\sqrt{t^2+1}} e^{\sqrt{t^2+1}}$

Solution

$$\frac{d}{d\sqrt{t^2+1}} e^{\sqrt{t^2+1}} = e^{\sqrt{t^2+1}}$$

Using the Decision Tree for Differentiation,
differentiation variable – t^2+1
function to differentiate - $e^{\sqrt{t^2+1}}$
function to differentiate is the table function - exponent - of the differentiation variable $\sqrt{t^2+1}$.

b) $\dfrac{d}{d\sqrt{t^2+1}} e^{\sqrt{x^2+1}}$

Solution

$$\frac{d}{d\sqrt{t^2+1}} e^{\sqrt{x^2+1}} = 0$$

Using the Decision Tree for Differentiation,
differentiation variable – t^2+1, it varies with *t*,
function to differentiate - $e^{\sqrt{x^2+1}}$.
This function does not contain *t*. Therefore it is a constant, which is a table function.

Q2. Find the derivatives

a) $\frac{d}{du}(u^2+1)\sin u$

Solution $\frac{d}{du}(u^2+1)\sin u = 2u\sin u + (u^2+1)\cos u$

Using the Decision Tree for Differentiation,
differentiation variable – u,
function to differentiate - $(u^2+1)\sin u$,
function to differentiate - not a table function (many ops on u).
The last operation is multiplication
Use the product rule $\frac{d(f(u)g(u))}{du} = \frac{df(u)}{du}g(u) + \frac{dg(u)}{du}f(u)$

f-line: The first factor $f(u) = u^2+1,\ \frac{df(u)}{du} = \frac{d}{du}(u^2+1) = 2u$

g-line: The second factor $g(u) = \sin u,\ \frac{dg(u)}{du} = \frac{d}{du}\sin u = \cos u$

Substitute $f(u)$, $\frac{df(u)}{du}$, $g(u)$ **and** $\frac{dg(u)}{du}$ **into the product rule.**

b) $\frac{d}{du}\frac{u^2+1}{\sin u}$

Solution $\frac{d}{du}\frac{u^2+1}{\sin u} = \frac{2u\sin u - (u^2+1)\cos u}{\sin^2 u}$

Using the Decision Tree for Differentiation,
differentiation variable – u,
function to differentiate - $\frac{u^2+1}{\sin u}$ **- not a table function.**
The last operation is division.
Use the quotient rule $\frac{d}{du}\frac{f(u)}{g(u)} = \frac{\frac{df}{du}g(u) - \frac{dg}{du}f(u)}{g^2}$

f-line: The numerator $f(u) = u^2+1,\ \frac{df(u)}{du} = \frac{d}{du}(u^2+1) = 2u$

g-line: The denominator $g(u) = \sin u,\ \frac{dg(u)}{du} = \frac{d}{du}\sin u = \cos u$

Substitute $f(u)$, $\frac{df(u)}{du}$, $g(u)$ **and** $\frac{dg(u)}{du}$ **into the quotient rule.**

Q3. Find the derivatives

a) $\frac{d\ \cos\ \sqrt{x^2+1}}{dx}$

Solution

$\frac{\mathrm{d}\cos\sqrt{\mathrm{x}^2+1}}{\mathrm{dx}} =$

Using Decision Tree for Differentiation,
differentiation variable – *x*,

function to differentiate - $\cos\left[(\mathrm{x}^2+1)^{1/2}\right]$.

Function to differentiate - not a table function
The last operation is cos (that is, other).
Use the chain rule (decompose, differentiate, multiply):

$$\frac{d(f(g(x)))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

	Decompose	Differentiate
f-line:	**Last operation** $\mathrm{f}(\,) = \cos(\,)$	$\frac{df(g)}{dg} = \frac{d\cos g}{dg} = -\sin g = -\sin\left[(\mathrm{x}^2+1)^{1/2}\right]$
g-line:	**Its argument** $\mathrm{g(x)} = (\mathrm{x}^2+1)^{1/2}$	$\frac{dg}{dx} = \frac{d}{dx}(\mathrm{x}^2+1)^{1/2}$

Multiply $\frac{df(g)}{dg}$ **and** $\frac{dg}{dx}$ **to obtain**

$$-\sin\left[(\mathrm{x}^2+1)^{1/2}\right]\frac{d}{dx}(\mathrm{x}^2+1)^{1/2} =$$

We have reduced the problem to a simpler one: now we have to differentiate the function $(\mathrm{x}^2+1)^{1/2}$.

Using Decision Tree for Differentiation,
differentiation variable – x,

function to differentiate - $(x^2+1)^{1/2}$.
Function to differentiate - not a table function.
The last operation is cos (that is, other).
Use the chain rule:

$$\frac{d(f(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

	Decompose	Differentiate
f- line:	The last operation $f() = ()^{1/2}$	$\frac{df(g)}{dg} = \frac{1}{2}g^{-1/2} = \frac{1}{2}(x^2+1)^{-1/2}$
g-line:	Its argument $g(x) = x^2+1,$	$\frac{dg}{dx} = 2x$

Multiply these new $\frac{df}{dg}$ and $\frac{dg}{dx}$ to obtain

$$-\sin\left[(x^2+1)^{1/2}\right]\frac{1}{2}(x^2+1)^{-1/2}2x = -\frac{x\sin\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

b) $\frac{d\cos\sqrt{x^2+1}}{dy}$

Solution

$$\frac{d\cos\sqrt{x^2+1}}{dy} = 0$$

Using the Decision Tree for Differentiation,
differentiation variable – y,

function to differentiate - $\cos\sqrt{x^2+1}$
This function does not contain y.
Therefore, it is a constant which is a table function.

Q3. Differentiate y with respect to t if $y = (t+\sqrt{t^2+a^2})^n$.

Solution

$$\frac{d}{dt}(t+\sqrt{t^2+a^2})^n =$$

Using the Decision Tree for Differentiation,
differentiation variable – t,

function to differentiate - $(t+\sqrt{t^2+a^2})^n$.
Function to differentiate is not a table function.
The last operation is power (that is, other).
Use the chain rule:

$$\frac{d(f(g(t))}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

f-line: The last operation $f(g) = g^n, \frac{df}{dg} = ng^{n-1} = n(t+\sqrt{t^2+a^2})^{n-1}$

g-line: Its argument $= t+\sqrt{t^2+a^2}$, $\frac{dg}{dt} = \frac{d}{dt}(t+\sqrt{t^2+a^2})$

Substitute $f(t)$, $\frac{df(t)}{dt}$, $g(t)$ **and** $\frac{dg(t)}{dt}$ **into the chain rule to obtain**

$$n(t+\sqrt{t^2+a^2})^{n-1}\frac{d}{dt}(t+\sqrt{t^2+a^2}) =$$

We have reduced the problem to a simpler one: now we have to differentiate the function $t+\sqrt{t^2+a^2}$.

Using the Decision Tree for Differentiation,
differentiation variable – t,

function to differentiate - $t+\sqrt{t^2+a^2}$.
Function to differentiate is not a table function.
The last operation is addition.
Use the "separate terms" rule.

$$n(t+\sqrt{t^2+a^2})(1+\frac{d}{dt}\sqrt{t^2+a^2}) =$$

We have reduced the problem to a simpler one: now we have to differentiate the function $\sqrt{t^2+a^2} = (t^2+a^2)^{1/2}$

Using the Decision Tree for Differentiation,
differentiation variable – t,
function to differentiate - $\sqrt{t^2+a^2}$.
Function to differentiate is not a table function.
The last operation is power.
Use the chain rule $\frac{d(f(g(t)))}{dt} = \frac{df}{dg}\frac{dg}{dt}$

f-line: The last operation $f(g) = g^{1/2}, \frac{df}{dg} = \frac{1}{2}g^{-1/2} = \frac{1}{2}(t^2+a^2)^{-1/2}$

g-line: Its argument $g(t) = t^2 + a^2, \frac{dg}{dt} = \frac{d}{dt}(t^2+a^2) = 2t$

Substitute these new $f(t)$, $\frac{df(t)}{dt}$, $g(t)$ **and** $\frac{dg(t)}{dt}$ **into the chain rule:**

$$n(t+\sqrt{t^2+a^2})^{n-1}\left(1+\frac{1}{2}(t^2+a^2)^{-1/2}2t\right) = n\,(t+\sqrt{t^2+a^2})^{n-1}\left(1+\frac{t}{\sqrt{t^2+a^2}}\right) =$$

Using the COMMON DENOMINATOR RULE to simplify the last factor, we get

$$n(t+\sqrt{t^2+a^2})^{n-1}\frac{\sqrt{t^2+a^2}+t}{\sqrt{t^2+a^2}} =$$

Using the algebraic rules The Order Of Terms Does Not Matter and Product Of Powers Is Power Of Products this simplifies to

$$n\frac{(t+\sqrt{t^2+a^2})^n}{\sqrt{t^2+a^2}}$$

Comments

Some students learnt to differentiate at school. If your answers are correct there is no problem. If they are not, rely on the Decision Tree. When differentiating you do not need to write as much as written above. Just indicate the main steps.

Of course, all differentiation methods are equivalent. The technique outlined here emphasises that at each step, **the last operation in the function dictates which differentiation rule to apply.** All you have to do after that is **substitute** the appropriate expressions into the appropriate rule.

Any problems with the Decision Tree - revise Lecture 4 on Functions and **Order Of Operations Summary**.

Solutions to Exercises in Lecture 16

Q1.
a) Sketch $y = -3\,(x^3 - 3x^2 + x)$ using analysis.

Solution

Step 1.

a) Find the first derivative of y.

Using the Decision Tree for Differentiation, the differentiation variable is x **and the function to differentiate is** $-3(x^3 - 3x^2 + x)$;
In this function the last operation is multiplication and one of the factors (-3) **is a constant - use the "constant factor out" rule**

$$\frac{dy}{dx} = -3\frac{d}{dx}(x^3 - 3x^2 + x) =$$

Using the Decision Tree for Differentiation, the function to differentiate is $x^3 - 3x^2 + x$, **in this function the last operation is addition - use the "separate terms" rule**

$$-3(\frac{d}{dx}x^3 - \frac{d}{dx}3x^2 + \frac{d}{dx}x) =$$

Using the Decision Tree for Differentiation,
in the first term the function to differentiation is x^3,
this is a table function – power
in the second term the function to differentiation is $3x^2$
in it the last operation is multiplication and one of the factors (3) **is a constant - use the "constant factor out" rule**
in the third term - the function to differentiation is x,
this is a table function - power

$$-3(3x^2 - 3\frac{d}{dx}x^2 + 1) =$$

Using the Decision Tree for Differentiation,
in the second term, the function to differentiation is x^2,
this is a table function - power

$$-3(3x^2 - 6x + 1)$$

b) Establish where the first derivative $\frac{dy}{dx} = -3(3x^2 - 6x + 1)$ is 0, >0 and <0.

To establish where it is 0 solve the equation $\frac{dy}{dx} = 0$, that is, the quadratic equation $3x^2 - 6x + 1 = 0$

Revision

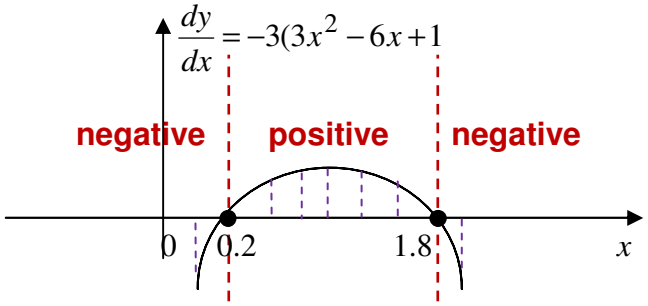
$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{6 \pm \sqrt{36 - 12}}{6} = \frac{6 \pm \sqrt{24}}{6} = \frac{6 \pm \sqrt{4 \cdot 6}}{6} = \frac{6 \pm 2\sqrt{6}}{6} = 1 \pm \frac{\sqrt{6}}{3}$$

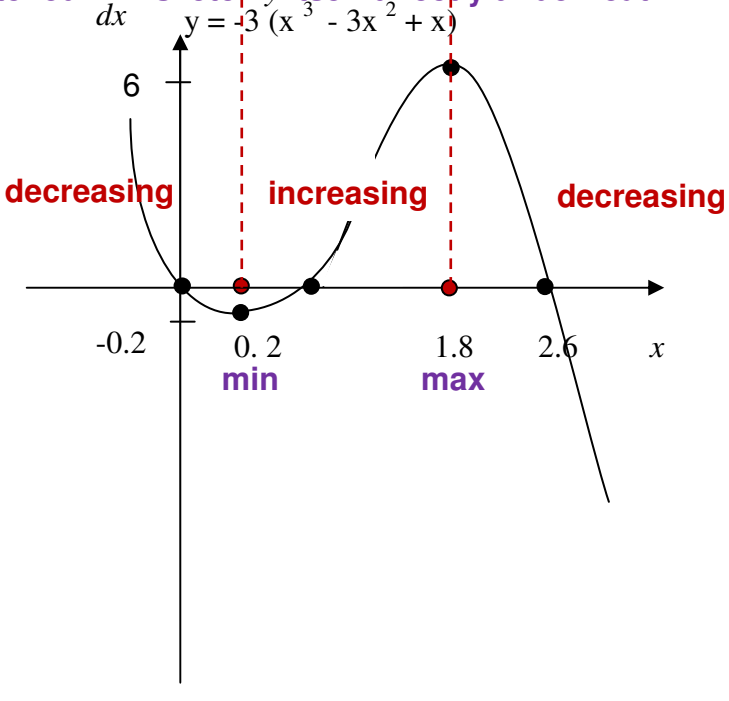
$$\approx 1 \pm 0.8 = 1.8, 0.$$

To establish where it is positive and where it is negative sketch the derivative. It is a parabola, with branches down (since the coefficient of the x^2 term is < 0)



Step 2.

a) Having sketched $\frac{dy}{dx}$ sketch y itself directly underneath.



b) Indicate on the graph the coordinates of special points

Intersection with y-axis:

$x = 0,\ y(0) = 0$

Intersection with x-axis:

$y = 0, \quad 0 = -3\,(x^3 - 3x^2 + x)$

$-3x\,(x^2 - 3x + 1) = 0$

$x_1 = 0$ **or** $x^2 - 3x + 1 = 0$

Revision

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{2,3} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} \approx \frac{3 \pm 2.2}{2} = 2.6,\ 0.4$$

Minimum:

$$y(0.2) = -3(0.2^3 - 3 \cdot 0.2^2 + 0.2) = -3 \cdot 0.2(0.2^2 - 3 \cdot 0.2 + 1) = -0.6(0.04 - 0.6 + 1) =$$
$$-0.6(-0.64 + 1) = -0.6(0.36) \approx -0.22$$

Maximum:

$$y(1.8) = -3(1.8^3 - 3 \cdot 1.8^2 + 1.8) = -3 \cdot 1.8(1.8^2 - 3 \cdot 1.8 + 1) = -5.4[1.8(1.8 - 3) + 1] =$$
$$-5.4[1.8 \cdot (-1.2) + 1] = -5.4(-2.16 + 1) \approx 6$$

b) $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Step 1.

a) Find the derivative of y**:** $\quad \frac{dy}{dx} = -\frac{1}{2} \cdot 2x + \frac{1}{4} = -x + \frac{1}{4}$

b) Establish where the first derivative $\frac{dy}{dx} = -x + \frac{1}{4}$ **is 0, positive and negative**

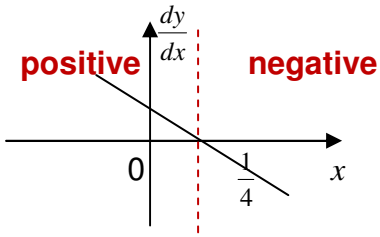
To establish where it is zero solve the equation

$$\frac{dy}{dx} = 0$$

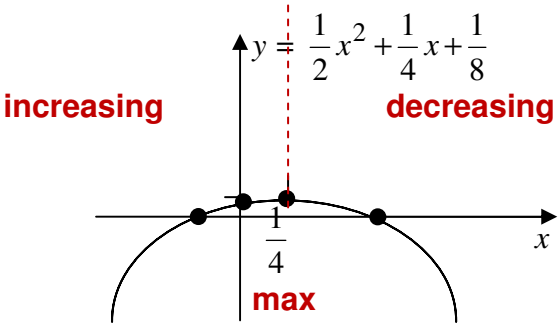
$$-x + \frac{1}{4} = 0$$

$$x = \frac{1}{4}$$

To establish where it is positive and where it is negative sketch the derivative. It is a straight line with a negative slope.



Step 2. a) Having sketched $\frac{dy}{dx}$ sketch y itself directly underneath.



b) Indicate on graph coordinates of special points

Intersection with y-axis:

$$x=0:\quad y=-\frac{1}{2}\cdot 0^2+\frac{1}{4}\cdot 0+\frac{1}{8}=\frac{1}{8}$$

Note that $\frac{1}{8}=\frac{4}{32}<\frac{5}{32}$, so the intersection with y-axis is below the tip

Intersection with x-axis:

$$y=0:\quad 0=-\frac{1}{2}x^2+\frac{1}{4}x+\frac{1}{8}\quad /\cdot(-8)$$

$$x_{1,2}=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 4\cdot(-1)}}{2\cdot 4}=\frac{2\pm\sqrt{4+4\cdot 4\cdot 1}}{8}=\frac{2\pm\sqrt{20}}{8}=\frac{2\pm\sqrt{4\cdot 5}}{8}=$$

$$\frac{1\pm\sqrt{5}}{4}\approx\frac{1}{4}\pm\frac{2.2}{4}=0.8,\quad -0.3$$

Maximum:

$$y=-y(\frac{1}{4})=-\frac{1}{2}(\frac{1}{4})^2+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{8}=-\frac{1}{2}\frac{1}{16}+\frac{1}{16}+\frac{1}{8}=-\frac{1}{32}+\frac{1}{16}+\frac{1}{8}=\frac{-1+2+4}{32}=\frac{5}{32}$$

Comments

When sketching graphs indicate coordinates of all special points (intersections with axes and maxima and minima) on the final graph.

Q3. Sketch $y = \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Solution

Step 1. Domain:

All reals apart from the values of x, such that $x^2 + 1 = 0$

$$x^2 = -1$$

This equation has no real solutions

Therefore, domain is "all reals without exceptions".

Therefore, there are no vertical asymptotes.

Step 2.

Intersection with the y-axis

$$x=0,\ y = \frac{0^2 - 5\cdot 0 + 6}{0^2 + 1} = 6$$

Intersection with the x-axis

$$y = 0,\ 0 = \frac{x^2 - 5x + 6}{x^2 + 1}$$

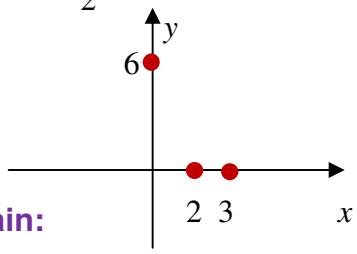
$$x^2 - 5x + 6 = 0$$

Revision

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

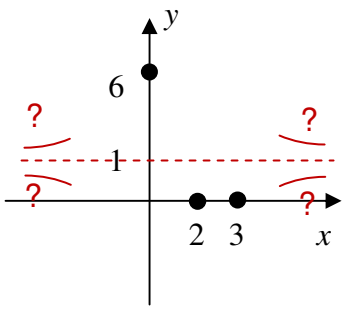
$$x_{1,2} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 1\cdot 6}}{2\cdot 1} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm \sqrt{1}}{2} \approx \frac{5 \pm 1}{2} = 3,2$$



Step 3. Behaviour at the outer boundaries of the domain:

$$\lim_{x\to\pm\infty} \frac{x^2 - 5x + 6}{x^2 + 1} = \lim_{x\to\pm\infty} \frac{1 - \frac{5}{x} + \frac{6}{x^2}}{1 + \frac{1}{x^2}} = \frac{1 - 0 + 0}{1 + 0} = 1 \qquad (\div x^2)$$

There is a horizontal asymptote $y = 1$



Step 4. a) Find the first derivative:

$$\frac{dy}{dx}=\frac{(2x-5)(x^2+1)-2x(x^2-5x+6)}{(x^2+1)^2}=\frac{2x^3+2x-5x^2-5-2x^3+10x^2-12x}{(x^2+1)^2}=$$

$$\frac{5x^2-10x-5}{(x^2+1)^2}=\frac{5}{(x^2+1)^2}(x^2-2x-1)$$

$$\frac{d\,\dfrac{f(x)}{g(x)}}{dx}=\frac{\dfrac{df}{dx}g(x)-\dfrac{dg}{dx}f(x)}{g^2}$$ – **quotient rule**

f-line: $f(x)=x^2-5x+6,\ \dfrac{df}{dx}=2x-5$

g-line: $g(x)=x^2+1,\ \dfrac{dg}{dx}=2x$

b) Establish where the first derivative $\dfrac{dy}{dx}=\dfrac{5}{(x^2+1)^2}(x^2-2x-1)$ is 0, >0 and <0.

The derivative is a product of two factors. The first one is always positive. Hence only the second one, x^2-2x-1, determines where the derivative is 0, >0 and <0. To establish where it is 0 solve the equation $\dfrac{dy}{dx}=0$, that is, the quadratic equation $x^2-2x-1=0$

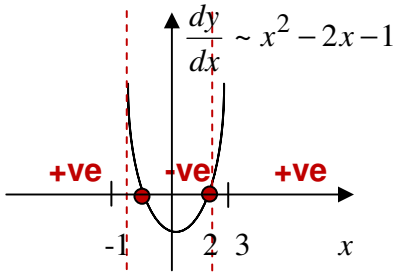
Revision

$$ax^2+bx+c=0$$

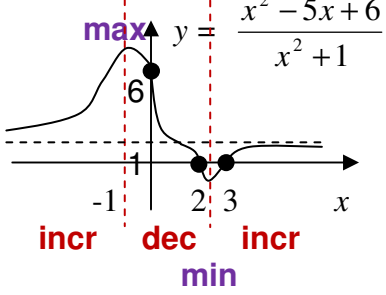
$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x_{1,2}=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot(-1)}}{2\cdot1}=\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm\sqrt{2\cdot4}}{2}\approx\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}\approx2.4,-0.4$$

To establish where it is positive and where it is negative sketch x^2-2x-1. It is a parabola with the branches up.



c) Having sketched the derivative sketch the function itself directly underneath.



d) Indicate on the graph the coordinates of maximum and minimum

$$y(1+\sqrt{2}) = \frac{(1+\sqrt{2})^2 - 5(1+\sqrt{2}) + 6}{(1+\sqrt{2})^2 + 1} = \frac{1+2\sqrt{2}+2-5-5\sqrt{2}+6}{1+2\sqrt{2}+2+1} = \frac{4-3\sqrt{2}}{4+2\sqrt{2}} \approx \frac{4-3\cdot 1.4}{4+2\cdot 1.4} \approx -0.7$$

$$y(1-\sqrt{2}) = \frac{(1-\sqrt{2})^2 - 5(1-\sqrt{2}) + 6}{(1-\sqrt{2})^2 + 1} = \frac{1-2\sqrt{2}+2-5+5\sqrt{2}+6}{1-2\sqrt{2}+2+1} = \frac{4+3\sqrt{2}}{4-2\sqrt{2}} \approx \frac{4+3\cdot 1.4}{4-2\cdot 1.4} = \frac{8.2}{1.2} \approx 6.5$$

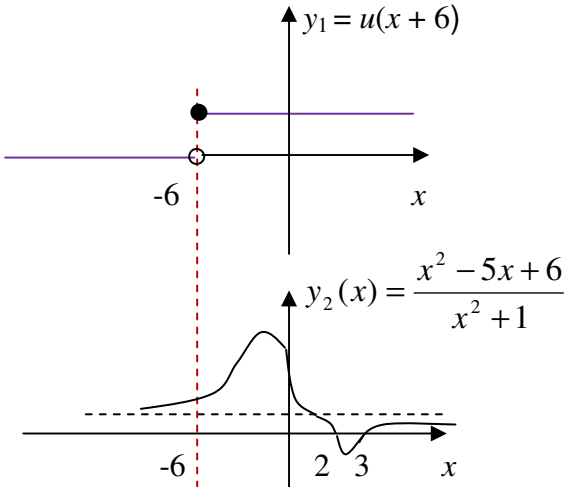
Q4. Sketch $y = u(x+2)\,\dfrac{x^2-5x+6}{x^2+1}$

Solution

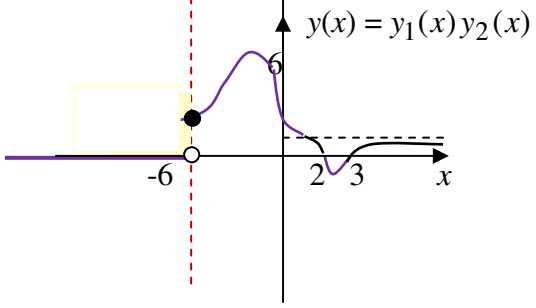
Step 1. Decompose given function into simpler functions that you can sketch:

$y(x) = y_1(x)y_2(x)$, **where** $y_1(x) = u(x+2), \quad y_2(x) = \dfrac{x^2-5x+6}{x^2+1}$

Step 2. a) Sketch functions $y_1(x)$ and $y_2(x)$ directly underneath one another



b) Sketch $y = y_1y_2$ by pointwise multiplication

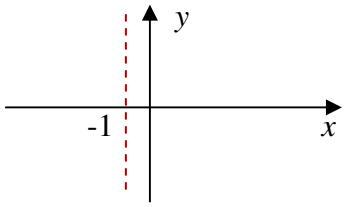


Q5. Sketch y = $\frac{x^2-5x+6}{x+1}$ **(advanced)**

Solution

Step 1. Domain:
All reals apart from values of x, such that $x+1=0$
$x=-1$

Therefore, $x \neq -1$
Therefore, there is a vertical asymptote $x=-1$



Step 2. Intersection with the y-axis:

$x=0,\ y=\frac{0^2-5\cdot 0+6}{0+1}=6$

Intersection with the x-axis:

$y=0,\ 0=\frac{x^2-5x+6}{x+1}$

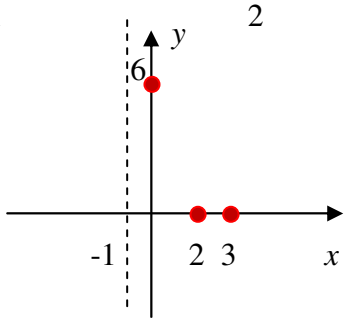
$x^2-5x+6=0$

Revision

$ax^2+bx+c=0$

$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x_{1,2}=\frac{-(-5)\pm\sqrt{(-5)^2-4\cdot 1\cdot 6}}{2\cdot 1}=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm\sqrt{1}}{2}\approx\frac{5\pm 1}{2}=3,2$



Step 3. Behaviour at the outer boundaries of the domain:

$\frac{x^2-5x+6}{x+1}$ **is an improper rational function**

The polynomial in the numerator is of degree 2**, the polynomial in the denominator is of degree** 1**. This means that the rational function** $\frac{x^2-5x+6}{x+1}$ **can be represented as a general polynomial of degree** $2-1=1$ **and a proper**

rational function with the same denominator:

$$\frac{x^2-5x+6}{x+1} = Ax+B+\frac{C}{x+1} =$$

To solve this equation for A, B **and** C **use the common denominator method:**

$$\frac{Ax+B}{1}\overset{(x+1)}{+}\frac{C}{x+1} = \frac{(Ax+B)(x+1)+C}{x+1}$$

Therefore,

$$\frac{x^2-5x+6}{x+1} = \frac{(Ax+B)(x+1)+C}{x+1}$$

Therefore,

$$x^2-5x+6 = (Ax+B)(x+1)+C =$$

Remove brackets

$$Ax^2+Ax+Bx+B+C =$$

Collect the like terms

$$Ax^2+(A+B)x+B+C$$

Equate the like coefficients

x^2-terms: $1 = A$

x-terms: $-5 = A + B \Rightarrow -5 = 1+B \Rightarrow B = -6$

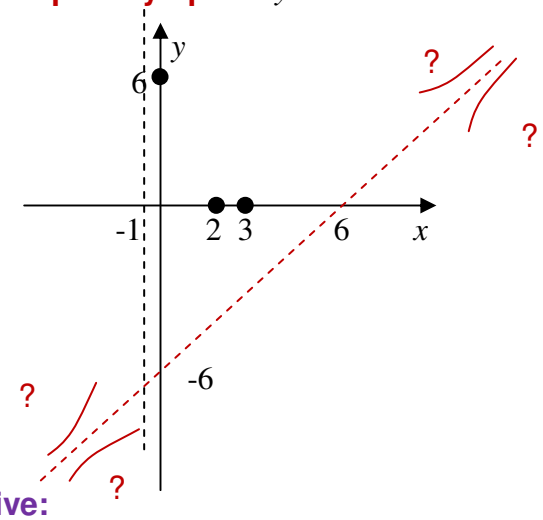
x^0-terms: $6 = B + C \Rightarrow 6 = -6+C \Rightarrow C = 12$

Therefore,

$$\frac{x^2-5x+6}{x+1} = x-6+\frac{12}{x+1} \sim x-6 \text{ as } x \to \pm\infty$$

Therefore, at infinity the function behaves as $y = x - 6$.

Therefore, there is an oblique asymptote $y = x - 6$



Step 4

a) Find the first derivative:

$$\frac{dy}{dx} = \frac{(2x-5)(x+1)-(x^2-5x+6)}{(x+1)^2} = \frac{2x^2+2x-5x-5-x^2+5x-6}{(x^2+1)^2} =$$

$$\frac{x^2+2x-11}{(x+1)^2} = \frac{1}{(x+1)^2}(x^2+2x-11)$$

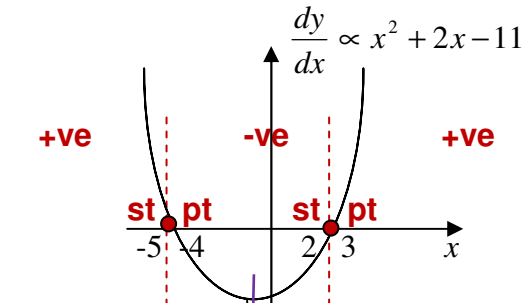
where we used the quotient rule $\dfrac{d\,\dfrac{f(x)}{g(x)}}{dx} = \dfrac{\dfrac{df}{dx}g(x)-\dfrac{dg}{dx}f(x)}{g^2}$

b) Establish where the first derivative $\frac{dy}{dx} = \frac{1}{(x+1)^2}(x^2+2x-11)$ is 0, >0 and <0.

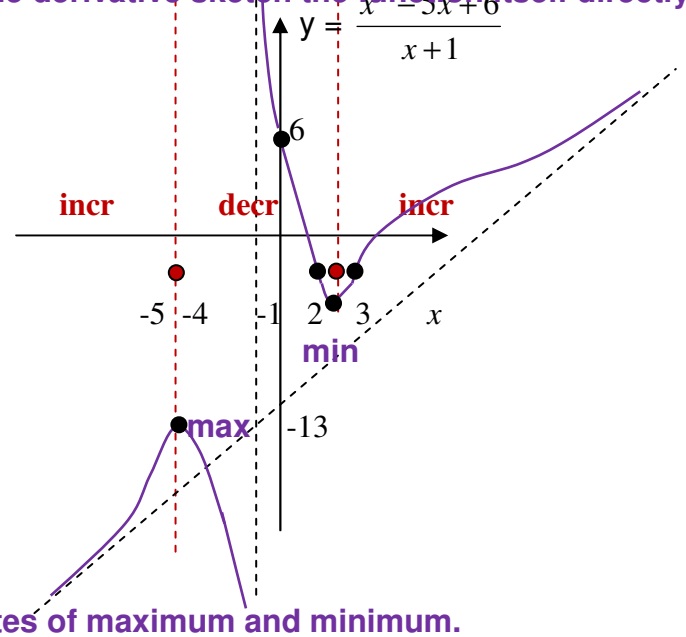
The derivative is a product of two factors. The first one is always positive. Therefore, only the second one, $x^2+2x-11$, determines where the derivative is 0, >0 and <0. To establish where it is zero solve the equation $\frac{dy}{dx}=0$, that is, the quadratic equation $x^2+2x-11=0$

$$x_{1,2} = \frac{-2\pm\sqrt{2^2-4\cdot1\cdot(-11)}}{2\cdot1} = \frac{-2\pm\sqrt{48}}{2} \approx \frac{-2\pm7}{2} = 2.5,-4.5$$

To establish where it is positive and where it is negative sketch $x^2+2x-11$. This is a parabola with the branches up.



c) Having sketched the derivative sketch the function itself directly underneath.



d) Find the coordinates of maximum and minimum.

$$y(2.5) = \frac{2.5^2-5\cdot2.5+6}{2.5+1} = \frac{6.25-12.5+6}{3.5} = \frac{-0.25}{3.5} \approx -0.07$$

$$y(-4.5) = \frac{(-4.5)^2-5\cdot(-4.5)+6}{-4.5+1} = \frac{20.25+22.5+6}{-3.5} = \frac{48.75}{-3.5} \approx -13$$

Solutions to Exercises in Lecture 17

Q1.

a) Find the first three non-zero terms in the Maclaurin series for $y=\frac{1}{1+x}$

Solution

Step 1. The general Maclaurin series is

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots$$

Step 2.	**Step 3.**
$f(x)=(1+x)^{-1}$ $f'(x)=-(1+x)^{-2}$ $f''(x)=2(1+x)^{-3}$	$f(0)=1$ $f'(0)=-1$ $f''(0)=2$

Step 4. $\frac{1}{1+x}=1-x+x^2+..$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\frac{1}{1+x}$ in the vicinity of $x=0$.

Solution

$$\frac{1}{1+x}\approx 1-x, |x|<1$$

c) Find, without using a calculator, the approximate value of $\frac{4}{5}$.

Solution

Let us express $\frac{4}{5}$ **as** $\frac{1}{1+x}$ **and solve the resulting equation for** x**:**

$$\frac{4}{5}=\frac{1}{1+x} \;/\cdot(1+x)$$

$$\frac{4}{5}(1+x)=1$$

Remove brackets

$$\frac{4}{5}+\frac{4}{5}x=1 \;/-\frac{4}{5}$$

$$\frac{4}{5}x=\frac{1}{5} \;/\cdot 5$$

$$4x=1\div 4$$

$$x=\frac{1}{4}$$

Using the approximation formula from c) above, $\frac{4}{5}=\frac{1}{1+\frac{1}{4}}\approx 1-\frac{1}{4}=\frac{3}{4}$

d) Find $\lim_{x\to -1}[\frac{1}{1+x}(1-e^{1+x})]$.

Solution

$\lim_{x\to -1}[\frac{1}{1+x}(1-e^{1+x})]=$

Substituting $x=-1$ get the $\frac{0}{0}$ indeterminate form. To evluate this limit use L'Hospital's rule

$$\lim_{x\to -1}[\frac{\frac{d}{dx}(1-e^{1+x})}{\frac{d}{dx}(1+x)}]=\lim_{x\to -1}[\frac{-e^{1+x}\frac{d}{dx}(1+x)}{\frac{d}{dx}(1+x)}]=\lim_{x\to -1}(-e^{1+x})=-1$$

Q2.

a) Find the first three non-zero terms in the Maclaurin series for $y=\frac{1}{1-x}$.

Solution

Step 1. The general Maclaurin series is

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+...$$

Step 2.	**Step 3.**
$f(x)=(1-x)^{-1}$ $f'(x)=(1-x)^{-2}$ $f''(x)=2(1-x)^{-3}$	$f(0)=1$ $f'(0)=1$ $f''(0)=2$

Step 4. $\frac{1}{1-x}=1+x+x^2+...$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\frac{1}{1-x}$ in the vicinity of x=0.

Solution

$\frac{1}{1-x}\approx 1+x, |x|<1$

e) Find, without using a calculator, the approximate value of $\frac{4}{3}$.

Solution

Let us express $\frac{4}{3}$ as $\frac{1}{1-x}$ and solve the resulting equation for x:

$$\frac{4}{3}=\frac{1}{1-x}\Big/(1\text{-}x)$$

$$\frac{4}{3}(1-x)=1$$

Remove brackets

$$\frac{4}{3}-\frac{4}{3}x=1\Big/-\frac{4}{3}$$

$$\frac{4}{3}x=\frac{1}{3}\Big/\cdot 3$$

$$4x=1\div 4$$

$$x=\frac{1}{4}$$

Using the approximation formula from c) above, $\frac{4}{3}=\frac{1}{1-\frac{1}{4}}\approx 1+\frac{1}{4}=\frac{5}{4}$

d) Find $\lim\limits_{x\to 1}[\frac{1}{1-x}\sin(1-x)]$.

Solution

$$\lim_{x\to 1}[\frac{1}{1-x}\sin(1-x)]=$$

Substituting $x=-1$ get the $\frac{0}{0}$ indeterminate form. To evluate this limit use L'Hospital's rule

$$\lim_{x\to 1}[\frac{\frac{d}{dx}\sin(1-x)}{\frac{d}{dx}(1-x)}]=\lim_{x\to 1}[\frac{\cos(1-x)\frac{d}{dx}(1-x)}{\frac{d}{dx}(1-x)}]=\lim_{x\to 1}\cos(1-x)=1$$

Q3.

a) Find the first three non-zero terms in the Maclaurin series for $y=\sqrt{1+x}$.

Solution

Step 1. The general Maclaurin series is $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots$

Step 2.	**Step 3.**
$f(x)=(1+x)^{1/2}$	$f(0)=1$
$f'(x)=\frac{1}{2}(1+x)^{-1/2}$	$f'(0)=\frac{1}{2}$
$f''(x)=-\frac{1}{4}(1+x)^{-3/2}$	$f''(0)=-\frac{1}{4}$

Step 4. $\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$

b) Use the first two terms of the above Maclaurin series to approximate $y=\sqrt{1+x}$ in the vicinity of $x=0$.

Solution

$\sqrt{1+x}\approx 1+\frac{1}{2}x, |x|<1$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{3}}$

Solution

Let us express $\frac{4}{3}$ **as** $1+x$ **and solve this for** x.

$\frac{4}{3}=1+x \;/\text{-}1 \Rightarrow x=\frac{1}{3}$

Using the approximation formula from c) above $\sqrt{\frac{4}{3}}=\sqrt{1+\frac{1}{3}}\approx 1+\frac{1}{2}\times\frac{1}{3}=1\frac{1}{6}$

d) Find $\lim\limits_{x\to -1}\frac{\sqrt{1+x}}{1-x^2}$.

Solution

$\lim\limits_{x\to -1}\frac{\sqrt{1+x}}{1-x^2}=$

Substituting $x=-1$ **the** $\frac{0}{0}$ **indeterminate form. To evluate this limit note that** $1+x\geq 0$ **(otherwise, there is no real solution) and use L'Hospital's rule**

$$\lim_{x\to -1}\frac{\frac{d}{dx}(1+x)^{1/2}}{\frac{d}{dx}(1-x^2)}=\lim_{x\to -1}\frac{\frac{1}{2}(1+x)^{-1/2}\frac{d}{dx}(1+x)}{-\frac{d}{dx}x^2}=\lim_{x\to -1}\frac{\frac{1}{2}(1+x)^{-1/2}}{-2x}=-\lim_{x\to -1}\frac{1}{4x\sqrt{1+x}}=\infty$$

Q4.

a) Find the first three non-zero terms in the Maclaurin series for $y=\sqrt{1-x}$

Solution

Step 1. The general Maclaurin series is $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots$

Step 2.	**Step 3.**
$f(x)=(1-x)^{1/2}$	$f(0)=1$
$f'(x)=-\frac{1}{2}(1-x)^{-1/2}$	$f'(0)=-\frac{1}{2}$
$f''(x)=-\frac{1}{4}(1-x)^{-3/2}$	$f''(0)=-\frac{1}{4}$

Step 4. $\sqrt{1-x}=1-\frac{1}{2}x-\frac{1}{8}x^2+\ldots$

b) Use the first two terms of the above Maclaurin series to approximate $y=\sqrt{1-x}$ in the vicinity of $x=0$.

Solution

$\sqrt{1-x}\approx 1-\frac{1}{2}x, |x|<1$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{5}}$

Solution

Let us express $\sqrt{\frac{4}{5}}$ **as** $\sqrt{1-x}$ **and solve this for** x.

$\frac{4}{5}=1-x \;/\times$ (-1) $\Rightarrow x=\frac{1}{5}$

Using the approximation formula from c) above $\sqrt{\frac{4}{5}}=\sqrt{1-\frac{1}{5}}\approx 1-\frac{1}{2}\times\frac{1}{5}=\frac{9}{10}$

d) Find $\lim\limits_{x\to 1}\frac{\sqrt{1-x}}{1-x^2}$.

Solution

$\lim\limits_{x\to 1}\frac{\sqrt{1-x}}{1-x^2}=$

Substituting $x=-1$ **get the** $\frac{0}{0}$ **indeterminate form. To evluate this limit we note that** $1-x\geq 0$ **(otherwise, there is no real solution) and use L'Hospital's rule**

$$\lim_{x\to 1}\frac{\frac{d}{dx}(1-x)^{1/2}}{\frac{d}{dx}(1-x^2)}=\lim_{x\to 1}\frac{\frac{1}{2}(1-x)^{-1/2}\frac{d}{dx}(1-x)}{\frac{d}{dx}(1)-\frac{d}{dx}x^2}=\lim_{x\to 1}\frac{-\frac{1}{2}(1-x)^{-1/2}}{-2x}=\lim_{x\to 1}\frac{1}{4x(1-x)^{1/2}}=\infty$$

Comments

The approximations in the first two problems are somewhat artificial ($0.8 \approx 0.75$ *etc.*) Nobody would use the Maclaurin series to do such approximations, the approach comes into its own only in Exercises 3 and 4.