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Differential Calculus Simplified to the Bone

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I. INTRODUCTION

These notes are based on the lectures delivered by the author to engineering students of London South Bank University over the period of 16 years. This is a University of widening participation, with students coming from many different countries, many of them not native English speakers. Most students have limited mathematical background and limited time both to revise the basics and to study new material. A system has been developed to assure efficient learning even under these challenging restrictions. The emphasis is on systematic presentation and explanation of basic abstract concepts. The technical jargon is reduced to the bare minimum.

Nothing gives a teacher a greater satisfaction than seeing a spark of understanding in the students' eyes and genuine pride and pleasure that follows such understanding. The author's belief that most people are capable of succeeding in - and therefore enjoying - the kind of mathematics that is taught at Universities has been confirmed many times by these subjective signs of success as well as genuine improvement in students' exam pass rates. Interestingly, no correlation had ever been found at the Department where the author worked between the students' qualification on entry and graduation.

The book owns a lot to the authors' students – too numerous to be named here - who talked to her at length about their difficulties and successes, e.g. see Appendix VII on Teaching Methodology. One former student has to be mentioned though – Richard Lunt – who put a lot of effort into making this book much more attractive than it would have been otherwise.

The author can be contacted through her website www.soundmathematics.com. All comments are welcome and teachers can obtain there the copy of notes with answers to questions suggested in the text as well as detailed Solutions to suggested Exercises. The teachers can then discuss those with students at the time of their convenience.

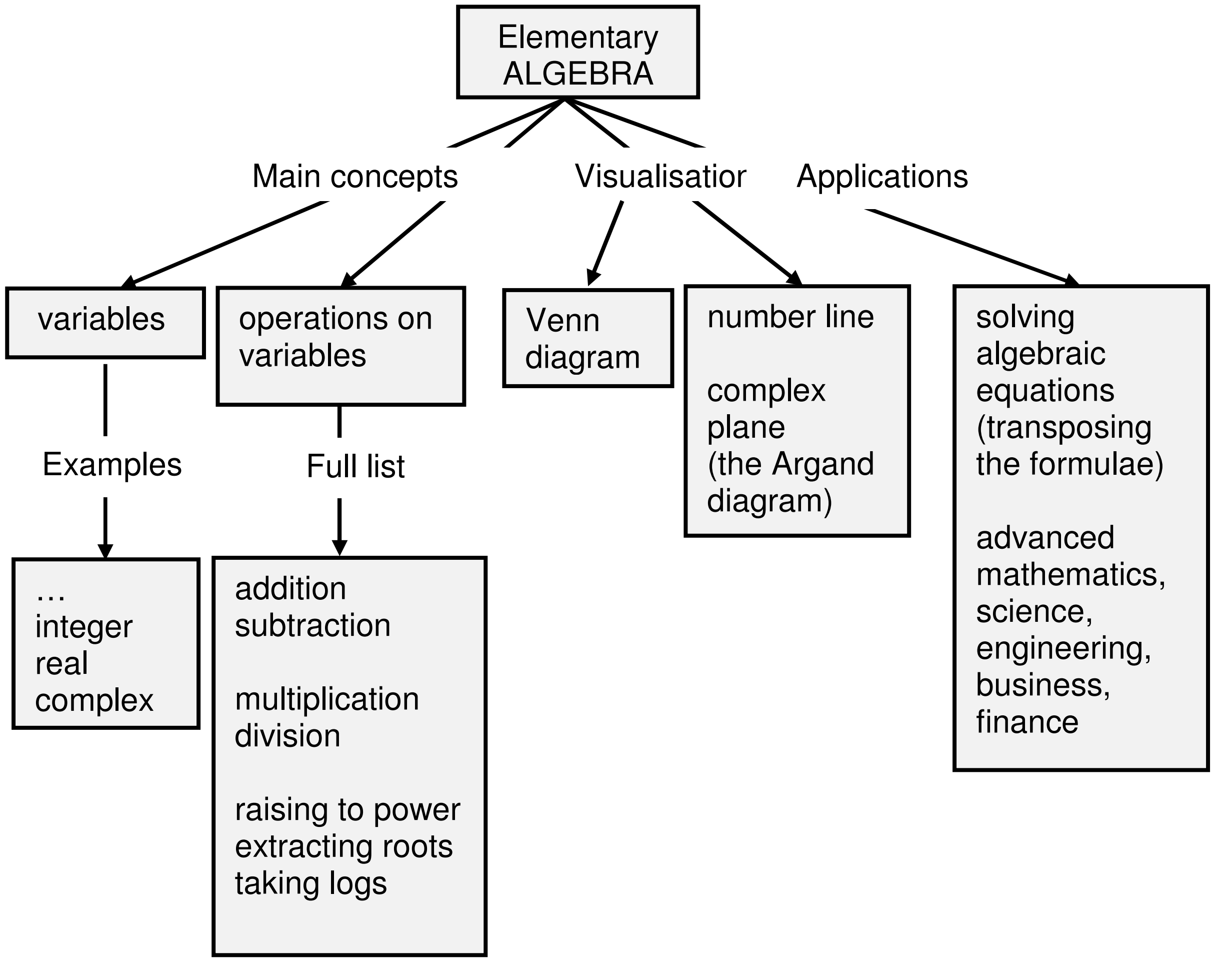
Good luck everyone!

II. CONCEPT MAPS

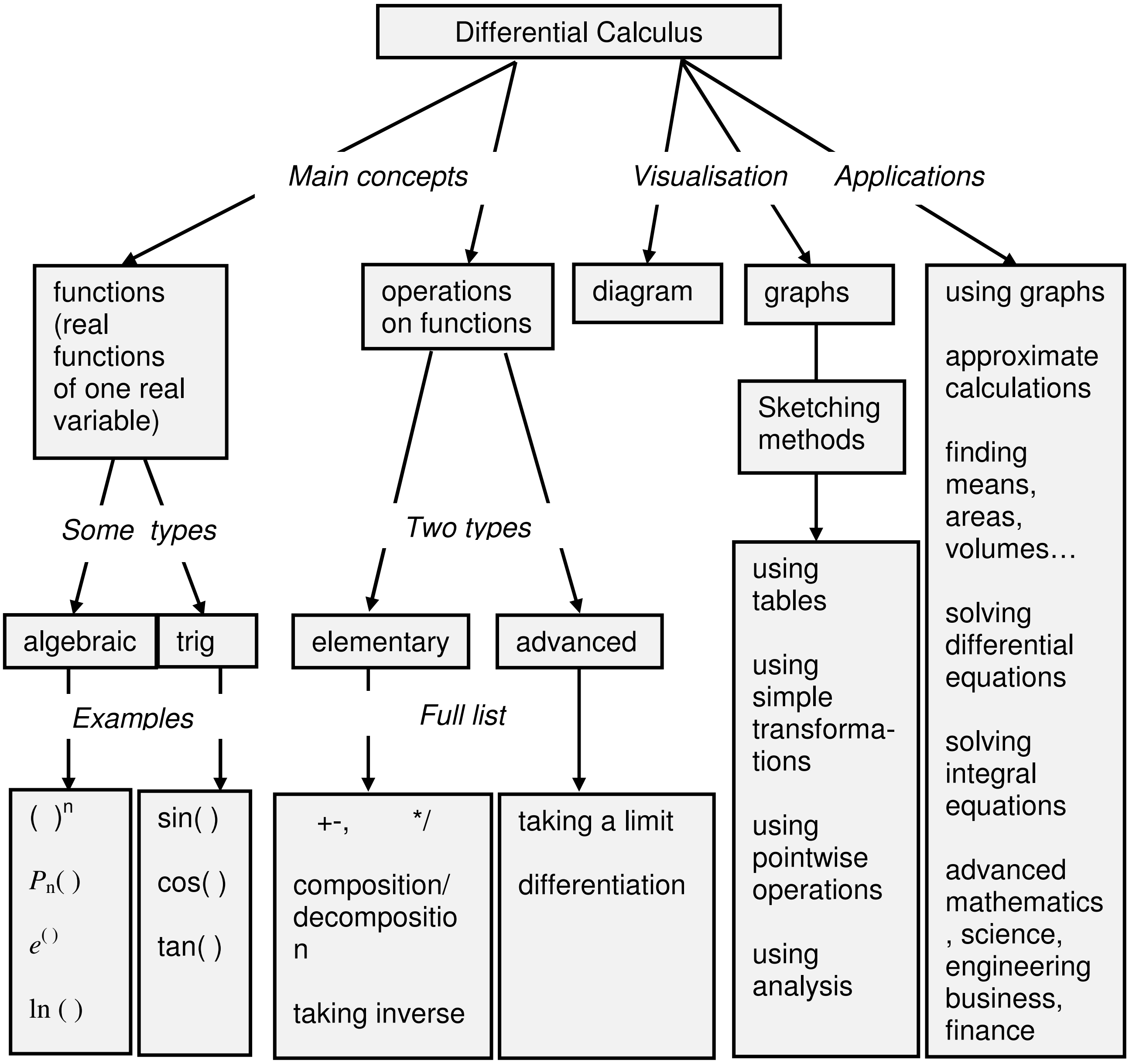
Throughout when we first introduce a new **concept** (a technical word or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea. Throughout when we first introduce a new concept (a technical word or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea.

One major topic is covered in these Notes, Differentiation Calculus. You can understand this topic best if you first study the Notes on Elementary Algebra and Functions.

Here is a **concept map** of Elementary Algebra.



Below is a concept map of Differential Calculus. It is best to study it before studying any of the Calculus Lectures to understand where it is on the map. The more you see of the big picture the better you learn!



III. LECTURES

Lecture 13. CALCULUS: Sequences, Limits and Series

In Calculus we study **functions and operations on functions** (see Calculus Concept Map). Elementary operations on functions (algebraic operations and composition/decomposition) were covered in Lectures 4 - 7. We now introduce the first **advanced operation on functions** – **taking a limit**. We introduce it first for discrete functions called sequences.

13.1 Sequences

A **sequence** is an ordered set of numbers x_n, a **discrete function** $x(n)$, where the argument is an integer n. This means that a discrete function is a function whose domain is the set of all integers or else its subset. Such argument is called the **counter**.

There are different ways of writing up a sequence: $\{x_n\}$ or $x_1, x_2, x_3, ..., x_N$ (each symbol n, 1, 2, 3, …, N can also be called an **index** or subscript). A sequence can be **finite** (having a fixed number of elements) or **infinite** (whatever index you specify there is always an element in the sequence with a greater index).

13.1.1. Defining a Sequence as Function of Counter

A sequence can be defined (described) by specifying a relationship between each element and its counter.

Examples:

1. Describe sequence $1, 2, 3, 4, ...$ using the above notations.

Solution

$x_1 = 1, x_2 = 2, x_3 = 3, ...$

Question: What is the relationship between the counter and the sequence element?

Answer:

$\Rightarrow \quad x_n = n$

2. Describe sequence $1, 4, 9, 16, ...$ using the above notations.

Solution

$x_1 = 1, x_2 = 4, x_3 = 9, ...$

Question: What is the relationship between the counter and the sequence element?

Answer:

$\Rightarrow x_n = n^2$

13.1.2. Visualising a Sequence as a Function of Counter

If a sequence is defined (described) by specifying a relationship between each element and its counter it can be visualised as a sequence of points on the number line (see figure 13.1) or else as a discrete graph (see figure 13.2).

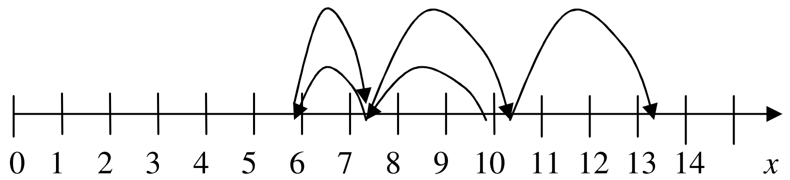


Figure 13.1. An example of a sequence x_n, $n = 0, 1, \ldots, 5$ visualised using the number line. The arrowed arcs are used to indicate the order of elements in the sequence.

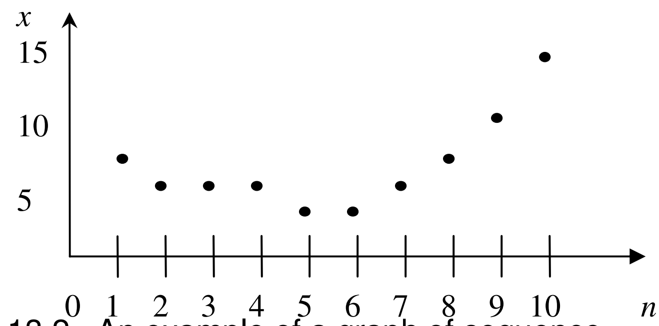


Figure 13.2. An example of a graph of sequence x_n, $n = 0, 1, \ldots, 10$.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

13.1.3 Applications of Sequences

A digital signal x (see figure 13.3 below) has the following characteristics:

1) it holds a fixed value for a specific length of time

2) it has sharp, abrupt changes

3) a preset number of values is allowed.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

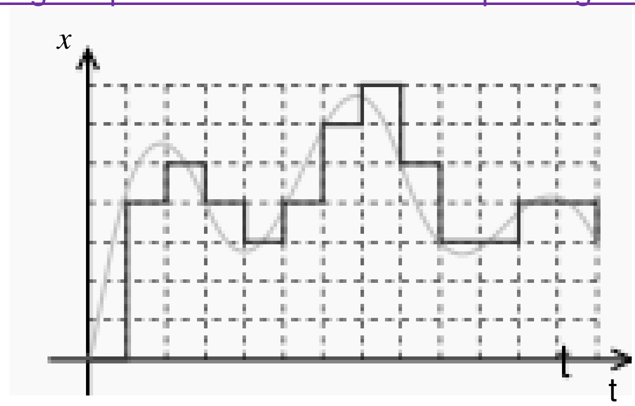


Figure 13.3. A typical digital signal. **http://www.privateline.com/manual/threeA.htm**
Therefore, digital signals are examples of sequences.

13.1.4. Defining a Sequence via a Recurrence Relationship

A sequence element may be defined in another way, via a **recurrence relation**

$$x_{n+1} = f(x_n, x_{n-1}, \ldots., x_1),$$

where f may be a function of several arguments and not just one argument.

Thus, there are two ways to describe a sequence,

1. using a functional relation $x(n)$, which specifies how each sequence element is defined by its counter;

2. using a recurrence relation $x_{n+1} = f(x_n, x_{n-1}, ..., x_1)$, which specifies how each sequence element is defined by previous sequence element(s).

Examples:

1. Given a sequence $x_n = n^2$ we can change the functional description to a recurrence relation

   $x_{n+1} = (n+1)^2 = n^2 + 2n + 1 \quad \Rightarrow \quad x_{n+1} = x_n + 2\sqrt{x_n} + 1$

   When given such a recurrence the first element needs to be specified. Only then can we start evaluating other elements.

2. A **Fibonacci sequence**: $1, 1, 2, 3, 5, 8, 13, 21, ...$ can be described via a recurrence relation

$$x_{n+1} = x_n + x_{n-1}.$$

When given such a recurrence the first two elements have to be specified. Only then can we start evaluating other elements. Let us check that the above recurrence describes the given sequence:

Question: What are x_1 and x_2?
Answer:

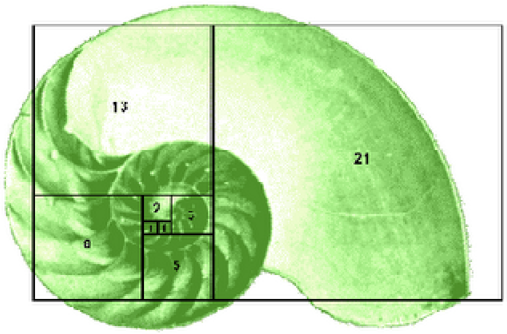
Question: Does x_3 satisfy the given recurrence relation and why?
Answer:

Question: Does x_4 satisfy the given recurrence relation and why?
Answer:

Question: Does x_5 satisfy the given recurrence relation and why?
Answer:

Fibonacci sequences in nature

When superimposed over the image of a nautilus shell we can see a Fibonacci sequence in nature:



http://munmathinnature.blogspot.com/2007/03/fibonacci-numbers.html

Each of the small spirals of broccoli below follows the Fibonacci's sequence.

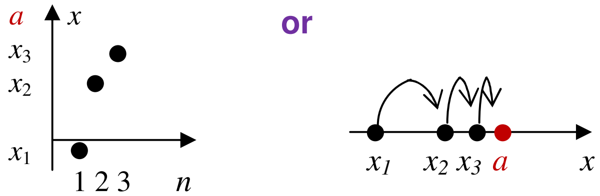
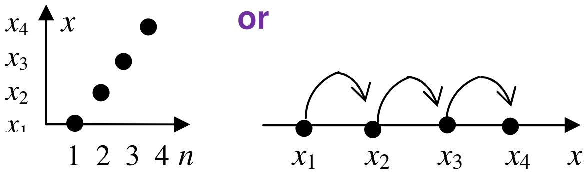
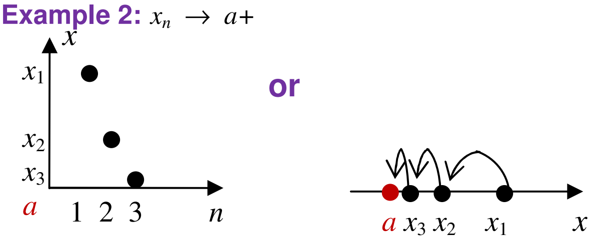
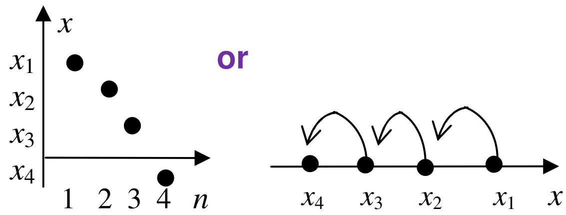
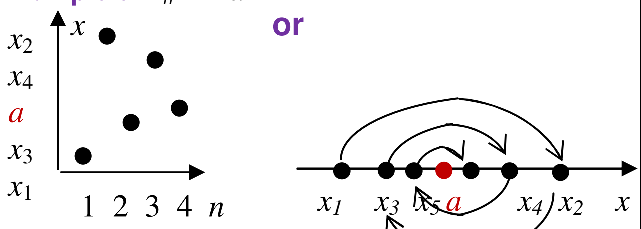
http://www.pdphoto.org/PictureDetail.php?mat=pdef&pg=8232

13.2 Limit of a sequence

Taking a limit of a sequence as n grows larger and larger without bounds is the first **advanced operation on function** that we cover. In mathematics, instead of the phrase n *grows larger and larger without bounds* we use the shorthand $n \to \infty$ (verbalised as n tends to infinity). If it exists the outcome of applying this operation to a (discrete) function x_n is called $\lim x_n$ and is either a number or else $\pm\infty$ (either +**infinity** or -**infinity**). Sometimes instead of $\lim x_n$ we write $\lim\limits_{n\to\infty} x_n$. However, the description $n \to \infty$ is usually understood and not mentioned.

Note: ∞ is not a number but a symbol of a specific sequence behavior, $+\infty$ means that the sequence increases without bounds and $-\infty$ means that the sequence decreases without bounds - see the right column in the Table in Section 13.2.1 below.

<u>13.2.1 Definition of a Limit of a Sequence</u>

Examples of a finite limit $\lim\limits_{(n\to\infty)} x_n = a$ (can also write $x_n \underset{(n\to\infty)}{\to} a$)	Examples of an infinite limit $\lim\limits_{(n\to\infty)} x_n = \pm\infty$ (can also write $x_n \underset{(n\to\infty)}{\to} \pm\infty$)
Example 1: $x_n \to a-$  x_n gets closer and closer to a from below and maybe reaches it.	Example 1: $x_n \to +\infty$  For any index N we can find a greater index n such that $x_n > x_N$ (sequence increases without bounds)
Example 2: $x_n \to a+$  x_n gets closer and closer to a from above and maybe reaches it.	Example 2: $x_n \to -\infty$  For any index N we can find a greater index n such that $x_n < x_N$ (sequence decreases without bounds)
Example 3: $x_n \to$ a  x_n gets closer and closer to a and maybe reaches it.	

Examples: Sketch the sequence to find its limit:

1. $x_n = \dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n come closer and closer to 0.

2. $x_n = -\dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n come closer and closer to 0.

3. $x_n = \dfrac{(-1)^{n+1}}{n} \to 0$ **Verbalise:** as n increases, x_n come closer and closer to 0.

4. $x_n = n \to \infty$ **Verbalise:** as n increases, sequence elements increase without bounds.

5. $x_n = -n \to -\infty$ **Verbalise:** as n increases, sequence elements decrease without bounds.

Sometimes, limits can be found using analytical considerations and sometimes, using the **Table**,
Rules and a **Decision Tree for Limits**:

13.2.2 Table of Limits

x_n	$\lim x_n$
$1/n$	0
$a^{1/n},\ a>0$	1
q^n, $\lvert q\rvert < 1$	0
$(1+\frac{1}{n})^n$	e

easy to find (first two rows)

the proof is rather involved (last two rows)

If $\{x_n\}$ has a finite limit the sequence is said to **converge**.

13.2.3 Rules for (Properties of) Limits

1. The limit is uniqie.

2. $\lim(\alpha x_n + \beta y_n) = \alpha \lim x_n + \beta \lim y_n,\ \ \alpha, \beta$ – constant
Verbalise: limit of a sum is a sum of limits and any constant factor can be taken out – **linearity property.**

3. $\lim x_n y_n = \lim x_n \lim y_n$
Verbalise: limit of a product is a product of limits.

4. $\lim \dfrac{x_n}{y_n} = \dfrac{\lim x_n}{\lim y_n}$, $\lim y_n \neq 0$
Verbalise: limit of a ratio is a ratio of limits.

Laws 2 - 4 give a general recipe for evaluating a limit of a composition of sequences – **instead of the sequence element substitute its limit.** That is, the Laws 2 – 4 state that if a limit exists the operation of taking the limit and any algebraic operation can be performed in an arbitrary order.

Optional

5. $x_n \geq y_n \Rightarrow \lim x_n \geq \lim y_n$ - **every element of one sequence is greater than or equal to the element with the same index in the other sequence** $\Rightarrow$ **its limit is also greater than or equal to the limit of the other sequence.**

6. $cx_n \leq y_n \leq z_n \Rightarrow \lim x_n = \lim z_n \Rightarrow \lim y_n = \lim x_n$
a sequence squeezed between two others with the same limit has the same limit.

Example: $x_n = (-1)^{n+1} n$. $\lim x_n$?

Solution: $\lim x_n$ does not exist $\Leftarrow$ Rule 1

13.2.4 Indeterminancy

Sometimes using the above rules leads to an **indeterminacy** (no obvious answer) like

$$\frac{0}{0}, \frac{\infty}{\infty}, \ 0\cdot\infty, \ \infty-\infty, \ 0^0, \ 1^{\infty} \ etc.$$ - **MEMORISE**

Question: Why do these outcomes present no clear answer?
Answer:

An indeterminacy can often be **resolved**, i.e. a clear answer can be found by using tricks.

Examples:

$\frac{0}{0}\downarrow$ **use algebraic trick – FLIP RULE**

1. $x_n = \frac{1}{n^2}, \ y_n = \frac{1}{n} \Rightarrow \lim \frac{x_n}{y_n} = \lim \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim \frac{1}{n} = 0$

$\frac{0}{0}\downarrow$ **use algebraic trick – FLIP RULE**

2. $x_n = \frac{1}{n^2}, \ y_n = \frac{1}{n} \Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim n = \infty$

3. $x_n = n, \ y_n = n^2$

$\infty-\infty\downarrow$ **use algebraic trick - factorisation**

$\Rightarrow \lim(x_n - y_n) = \lim(n - n^2) = \lim n(1-n) = \lim n \lim(1-n) = \infty\cdot(-\infty) = -\infty$

4. $x_n = n^2, \ y_n = n^2 - 1$

$\frac{\infty}{\infty}\downarrow$ **use algebraic trick – divide top and bottom by the highest power**

5. $\Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{n^2-1}{n^2} = \lim \frac{n^2 - \cancel{1} \div n^2}{n^2} = \lim \frac{1-\frac{1}{n^2}}{1} = 1 - 0 = 1$

13.2.5 Decision Tree for Limits

Decision Tree (a Flow Chart) that can be helpful when evaluating limits is given in figure 13.1. Start reading it from the very top and follow the arrows as dictated by your answers. In many situations the tricks necessary to find a limit are very involved. However, in standard undergraduate courses they require only the use of simple algebraic and trigonometric laws and formulae

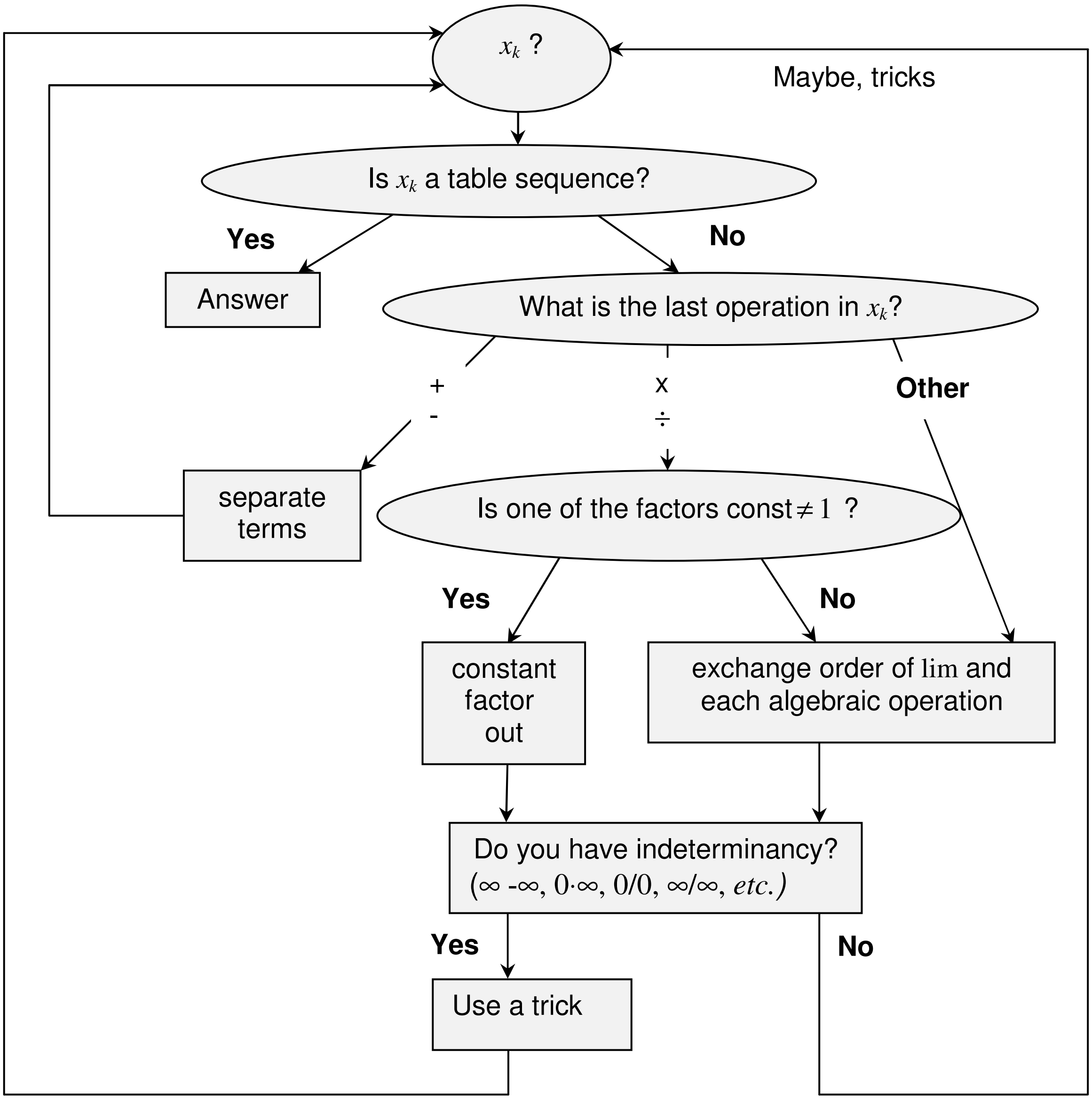


Figure 12.1. Decision Tree for Limits.

13.3 Series

A **series** is a sum of ordered terms (a sum of sequence elements),
Question: What is a sum?
Answer:

A series $x_1 + ... + x_N$ can be written using the shorthand S_N or $\sum_{n=1}^{N} x_n$. In the latter case we say that we use the **sigma notation**, because $\sum$ is a capital Greek letter called *sigma.*

13.3.1 Arithmetic Progression

An arithmetic progression is a sequence defined by the following recurrence relationship: $x_{n+1} = x_n + a$, where a is constant (with respect to n). a is called the **common difference.**

$$\Rightarrow S_N = x_1 + ... + x_N = N\frac{x_1 + x_N}{2}.$$ **MEMORISE**

13.3.2 Geometric Progression

A geometric progression is a sequence defined by the following recurrence relationship: $x_{n+1} = ax_n$, where where a is constant (with respect to n). a is called the **common ratio.**

$$\Rightarrow S_N = \frac{x_1(1-a^N)}{1-a}$$

If $|a| < 1$, then $S_N = \frac{x_1}{1-a} - \frac{x_1 a^N}{1-a} \to \frac{x_1}{1-a} \Rightarrow \lim S_N = \frac{x_1}{1-a}$. **MEMORISE**

13.4 Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and TRIGONOMETRY**
- **Revise Lecture 11 and study Solutions to Exercises in Lecture 11 using the STUDY SKILLS Appendix**
- **Revise Lecture 12 using the STUDY SKILLS Appendix**
- **Study Lecture 13 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find the following limits:

a) $\lim \frac{n^2+n+1}{2n+1}$;

b) $\lim \frac{n^2+n+1}{2n^2+1}$;

c) $\lim \frac{n^2+n+1}{2n^3+1}$

Q2. Find the sum of a hundred terms of the arithmetic series with first term 1 and common difference 2.

Q3. A steel ball-bearing drops on to a smooth hard surface from a height h. The time to first impact is $T = \sqrt{\frac{2h}{g}}$, where g is the acceleration due to gravity. The times between successive bounces are $2eT, 2e^2T, 2e^3T,...$, where e is the coefficient of restitution between the ball and the surface $(0 < e < 1)$. Find the total time taken up to the fifth bounce if $T = 1$ and $e = 0.1$.

Q4 (**advanced**). The irrational number π can be defined as the area of a unit circle (circle of radius 1). Thus, it can be defined as a limit of a sequence of known numbers a_n. The method used by Archimedes was to inscribe in the unit circle a sequence of regular polygons (that is, polygons with equal sides – see figure 8.1). As the number of sides increases the polygon "fills" the circle. For any given number n find the area a_n of the regular polygon inscribed in the unit circle. Show, by use of trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras theorem), that the area a_n satisfies the recurrence relation

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1 - \sqrt{\left[1 - \left(\frac{2a_n}{n}\right)^2\right]},\ n \geq 4.$$

Use trigonometry to show that $a_4 = 2$ and use the above recurrence relation to find a_8.

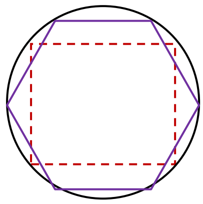


Figure 8.1. A circle with two inscribed regular polygons, one with four sides (a square) and one, with six (a hexagon).

Q5. Explain why $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$.

Lecture 14. DIFFERENTIAL CALCULUS: Limits, Continuity and Differentiation of Real Functions of One Real Variable

14.1 Limits

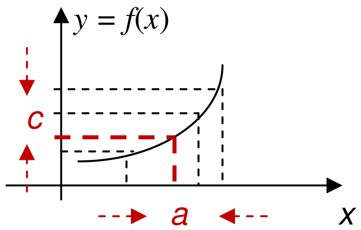
Taking a limit is **an advanced operation on functions**. The outcome of this operation is a number (more generally, a constant with respect to the control variable). The operation involves finding a limiting value of the dependent variable when the independent variable approaches (tends to) a specified limiting value of its own or else to ∞ or $-\infty$.

14.1.1 Definition of a Limit of a Real Function of One Real Variable

Four limiting behaviours are possible:

1. $\lim\limits_{x \to a} f(x) = c$

Verbalise: as x tends to a finite limit a, $f(x)$ tends to a finite limit c.



2. $\lim\limits_{x \to \infty} f(x) = c$ or $\lim\limits_{x \to -\infty} f(x) = c$

Verbalise: as x tends (goes off) to infinity, f(x) tends to a finite limit c.

3. $\lim\limits_{x \to \infty} f(x) = \infty$ or $-\infty$, $\lim\limits_{x \to -\infty} f(x) = \infty$ or $-\infty$,

Verbalise: as x tends (goes off) to + or - infinity, $f(x)$ tends to + or - infinity.

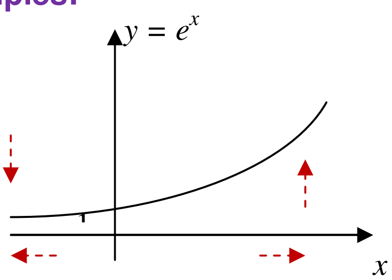
4. $\lim\limits_{x \to a} f(x) = \infty$ or $-\infty$

Verbalise: as x tends to (approaches) a finite limit a, $f(x)$ tends to + or - infinity.

Note: symbols $\pm\infty$ are not numbers, they describe a specific function behaviour.

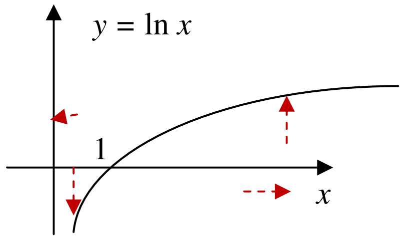
Examples:

1.



$\lim\limits_{x \to -\infty} e^x = 0$ $\qquad$ $\lim\limits_{x \to \infty} e^x = \infty$

2.



$\lim\limits_{x\to 0+} \ln x = -\infty$ $\qquad\qquad$ $\lim\limits_{x\to\infty} \ln x = \infty$

14.1.2 Table of Limits of Functions

$f(x)$	$\lim\limits_{x\to 0} f(x)$
$\dfrac{\sin x}{x}$	1
$\dfrac{e^x - 1}{x}$	1

the proof is rather involved

14.1.3 Rules for (Properties of) Limits

1. The limit is unique (that is, there can be only one limiting value).

2. $\lim\limits_{x\to a} [f_1 \begin{smallmatrix}+\\-\\\times\\\cdot\end{smallmatrix} f_2(\mathrm{x})] = \lim\limits_{x\to a} f_1(x) \begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix} \lim\limits_{x\to a} f_2(x)$ if all limits exist

Note: division is allowed only if $\lim\limits_{x\to a} f_2(x) \neq 0$

Using these rules is equivalent to just substituting a for x.

Optional

3. $f_1(x) \le f_2(x) \le f_3(\mathrm{x})$ and $\lim\limits_{x\to a} f_1(x) = \lim\limits_{x\to a} f_3(x) \Rightarrow \lim\limits_{x\to a} f_2(\mathrm{x}) = \lim\limits_{x\to a} f_1(x)$

Examples:

1. $\lim\limits_{x\to 1} (x^2 - 4x + 8) = 1^2 - 4\cdot 1 + 8 = 5$ **(instead of x we substituted its limiting value)**
2. $\lim\limits_{x\to \pi/2} \sin x = 1$ **(instead of x we substituted its limiting value)**
3. $\lim\limits_{x\to \pi/2} \cos x = 0$ **(instead of x we substituted its limiting value)**

4. $\lim_{x\to\pi/2} \tan x$ does not exist, but $\lim_{x\to\pi/2^{\pm}} \frac{\sin x}{\cos x} = \mp\infty$, which is a shorthand for two statements, $\lim_{x\to\pi/2+} \frac{\sin x}{\cos x} = -\infty$ and $\lim_{x\to\pi/2-} \frac{\sin x}{\cos x} = +\infty$

5. $\lim_{x\to 2} \overset{\frac{0}{0}}{\frac{x^2-4}{x-2}} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$

Verbalise: zero over zero indeterminacy – we resolve it using algebraic tricks.

Sometimes, limits can be found using analytical considerations and sometimes, using the **Table**, **Rules** and a **Decision Tree for Limits** (practically the same as the one for sequences).

14.1.4 Applications of the Concept of a Limit

1. "Imagine a person walking over a landscape represented by the graph of $y = f(x)$. Her horizontal position is measured by the value of x, much like the position given by a map of the land or by a global positioning system. Her altitude is given by the coordinate y. She is walking towards the horizontal position given by $x = a$. As she does so, she notices that her altitude approaches L. If later asked to guess the altitude over $x = a$, she would then answer L, even if she had never actually reached that position."
http://en.wikipedia.org/wiki/Limit_of_a_function#Motivation

2. It is often important for engineers to assess how a measured quantity (current, voltage, density, load) behaves with time: does it grow without bounds (then its limit is ∞), tends to a specific value (then its limit is a finite number), oscillates (then it has no limit) *etc.*

14.1.5 Existence of a Limit of a Function

A limit of a function does not always exist.

Examples:

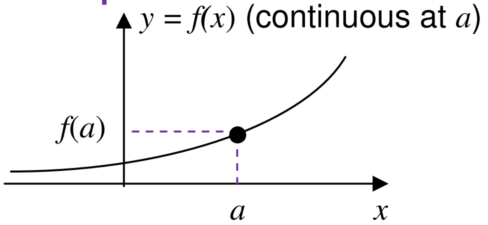
1. $\lim_{x\to\infty} \sin x$ - does not exist.

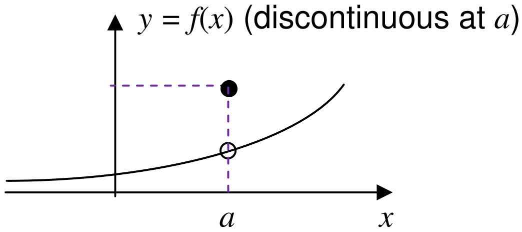
2. $\lim_{x\to\infty} \sin\frac{1}{x} = 0.$

14.2 Continuity of a function

A function $f(x)$ is said to be **continuous** at point a if $\lim_{x\to a} f(x) = f(a)$.

Examples:





Note: the filled circle indicates the point that belongs to the graph and an empty circle – the point that does not.

If a function is continuous at each point of an interval it is **continuous on** this **interval**. **Intuitively, a continuous function is a function for which, smaller and smaller changes in the input (argument or independent variable) result in smaller and smaller changes in the output (value of a function or dependent variable).** Otherwise, a function is said to be **discontinuous**.

If $f_1(x)$ and $f_2(x)$ are continuous on interval, so is $f_1(x) \begin{smallmatrix} + \\ - \\ \times \\ \div \end{smallmatrix} f_2(x)$ (provided $f_2(x) \neq 0$ when dividing).

14.2.1 Applications of Continuous Functions

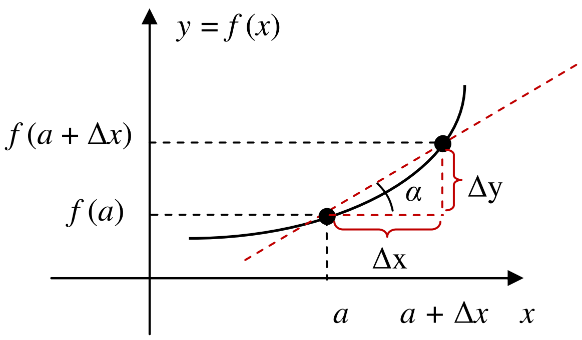
An analogue signal is a continuous signal for which the time varying feature of the signal is a representation of some other time varying quantity... For example, in sound recording, fluctuations in air pressure (that is to say, sound) strike the diaphragm of a microphone which causes corresponding fluctuations in a voltage or the current in an electric circuit. The voltage or the current is said to be an "analog" of the sound.
http://en.wikipedia.org/wiki/Analog_signal

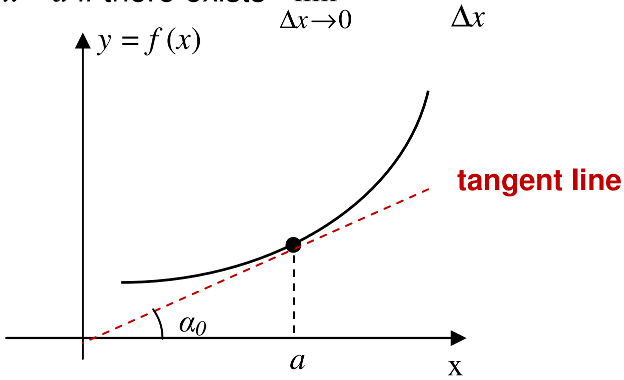
14.3 Differentiation of a real function of one real variable

Differentiation is the second advanced operation on functions we cover. The outcome of this operation is a function, which is called **the derivative** of the original function.

14.3.1 A Derivative of a Function

A continuous function $f(x)$ has a **derivative** at $x = a$ if there exists $\lim\limits_{\Delta x\to 0}\dfrac{f(a+\Delta x)-f(a)}{\Delta x} \overset{\frac{0}{0}}{=}$





$= \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}\tan\alpha = \tan\alpha_0 \equiv \dfrac{dy}{dx}(a) \equiv \dfrac{df}{dx}(a) \equiv f'(a)$ This limit is called a **derivative**.

Thus, we arrived at the following **definition of the derivative**:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

14.3.2 Geometrical Interpretation of a Derivative

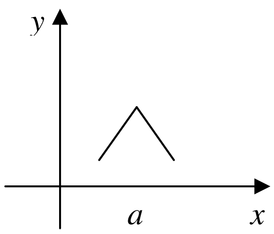
A derivative of a function is a local slope (gradient) of the function at a point. More specifically, it is the slope of the line tangent to the graph of the function at this point.

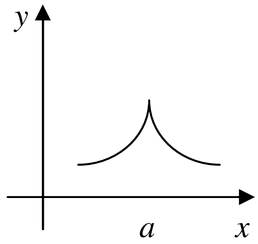
14.3.3 Mechanical Interpretation of a Derivative

A derivative of a function is a local rate (speed) of change of the function at a point.

14.3.4 Existence of a Derivative

A derivative of a function does not always exist.
Here are graphs of functions which have no derivative at point a of their domain:

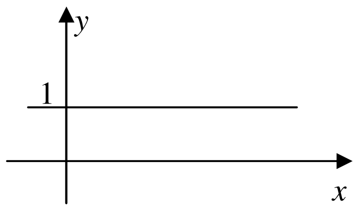




Examples:

1. Differentiate $y = c$ (constant)

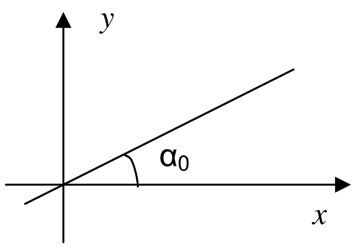
Can differentiate using a graph: the slope is the same everywhere and is zero.



$$\Rightarrow \frac{dy}{dx} = 0$$

2. $y = c\,x$

a) Can differentiate using a graph: the slope is the same everywhere – c



$$\Rightarrow \frac{dy}{dx} = c$$

b) Can differentiate using the basic definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c(x+\Delta x) - cx}{\Delta x} = c$$

14.4 A historical note

“When Newton and Leibniz first published their results on calculus in the 17the century, there was great controversy over which mathematician (and therefore which country) deserved credit. Newton derived his results first, but Leibniz published first. Newton claimed Leibniz stole ideas from his unpublished notes, which Newton had shared with a few members of the Royal Society. This controversy divided English-speaking mathematicians from continental mathematicians for many years, to the detriment of English mathematics. A careful examination of the papers of Leibniz and Newton shows that they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. Today, both Newton and Leibniz are given credit for developing calculus independently. It is Leibniz, however, who gave the new discipline its name. Newton called his calculus “the science of fluxions". **http://en.wikipedia.org/wiki/Calculus**

14.5 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 12 and study Solutions to Exercises in Lecture 12 using the STUDY SKILLS Appendix**
- **Revise Lecture 13 using the STUDY SKILLS Appendix**
- **Study Lecture 14 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (using $\lim_{x\to 0}\frac{\sin x}{x}=1$ and trigonometric identities)

b) $\lim_{x\to 0}\frac{e^x-e^{-x}}{2}$

c) $\lim_{x\to 0}\frac{\sin^2 3x}{x^3}$

d) $\lim_{x\to 0}\frac{x^2-1}{2x^2+3}$

e) $\lim_{x\to \infty}\frac{x^2-1}{2x^2+3}$

Q2. Discuss continuity of $y = \tan x$.

Q3. Plot $y = e^x$. Differentiate this function approximately using the graph and plot the approximate derivative $\frac{d}{dx}e^x$.

Q4. Prove

a) $\frac{d}{dx}x^n = nx^{n-1}$ (Hint: use the Binomial Theorem – look it up in books or on google)

b) $\frac{d}{dx}e^x = e^x$

c) $\frac{d}{dx}\sin x = \cos x$

d) $\frac{\mathrm{d}}{\mathrm{dx}}\cos \mathrm{x} = -\sin \mathrm{x}$

Lecture 15. DIFFERENTIAL CALCULUS: Differentiation (ctd.)

Using **the first principles** (that is, the definition of a derivative) we can create a **table of derivatives of elementary functions**.

15.1 Differentiation Table

function to differentiate

argument

differentiation variable

$y = f(x)$	$\dfrac{df(x)}{dx}$
const	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

This row means $\dfrac{d\,\text{const}}{dx} = 0$

This row means $\dfrac{d\,x^n}{dx} = nx^{n-1}$

This row means $\dfrac{d\,e^x}{dx} = e^x$

This row means $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$

This row means $\dfrac{d\sin x}{dx} = \cos x$

This row means $\dfrac{d\cos x}{dx} = -\sin x$

Examples:

1. $\dfrac{d\cos\pi}{dx} = 0$

Differentiation variable – x
Function to differentiate – constant
It is the table function

2. $\dfrac{d\,t^3}{dt} = 3t^2$

Differentiation variable – t
Function to differentiate – power 3
It is the table function of t

3. $\dfrac{d \sin v}{dv} = \cos v$

Differentiation variable – v
Function to differentiate – $\sin(\ )$
It is the table function of v

4. $\dfrac{d(\ln t)^3}{d(\ln t)} = 3(\ln t)^2$

Differentiation variable – $\ln t$
Function to differentiate – power 3
It is the table function of $\ln t$

5. $\dfrac{d \cos e^v}{de^v} = -\sin e^v$

Differentiation variable – e^v
Function to differentiate – $\cos(\ )$
It is the table function of e^v

To differentiate combinations of elementary functions we can use **Differentiation Table, Dfferentiation Rules** and **Decision Tree** given in figure 15.1.

15.2 Differentiation Rules

1. $\dfrac{d\alpha f(x)}{dx} = \alpha \dfrac{df(x)}{dx}$ - **constant factor out rule**
2. $\dfrac{d[f(x)+g(x)]}{dx} = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$ – **sum rule**

linearity property

3. $\dfrac{d(f(x)g(x))}{dx} = \dfrac{df(x)}{dx} g(x) + \dfrac{dg(x)}{dx} f(x)$ – **product rule**
4. $\dfrac{d\dfrac{f(x)}{g(x)}}{dx} = \dfrac{\dfrac{df(x)}{dx} g(x) - \dfrac{dg(x)}{dx} f(x)}{g^2}$ – **quotient rule**
5. $\dfrac{d(f(g(x)))}{dx} = \dfrac{df(g)}{dg} \dfrac{dg}{dx}$ – **chain rule (decompose, differentiate, multiply)**. Here $g = g(x)$

Note: These rules can be proven using the definition of the derivative as a limit and rules for limits.

Examples:

1. $\dfrac{d\ 2\ \sin\ x}{dx} = 2\cos\ x$

Differentiation variable – x
Function to differentiate – $2\sin x$
Last operation in this function – multiplication
One of the factors constant
Apply "constant factor out" rule

Constant factor $\alpha = 2$

Variable Factor $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

2. $\dfrac{d\,(\ \sin\ x\ + \cos\ x)}{dx} = \cos x -\ \sin x$

Differentiation variable – x
Function to differentiate – $3\sin x + 5\cos x$
Last operation in this function – addition
Apply the sum rule

---

f-line: 1st term $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

g-line: 2nd term $g(x) = \cos x$, $\dfrac{d\,g(x)}{dx} = \dfrac{d\ \cos\ x}{dx} = -\sin x$

3. $\dfrac{d\,(x^2 +\ 3x\ + 2\,)}{dx} = \dfrac{d\,x^2}{dx} + \dfrac{d\ 3x}{dx} + \dfrac{d\ 2}{dx} = 2x + 3$

Differentiation variable – x

Function to differentiate – $x^2 + 3x + 2$
Last operation in this function – addition
Apply the sum rule
Then go over the same reasoning for each term

4. $\dfrac{d\,(x \sin x)}{dx} = \sin x + x \cos x$

Last operation in this function – multiplication
Is one of the factors constant? No
Apply the product rule

f-line: 1st factor $f(x) = x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\,x}{dx} = 1$

g-line: 2nd factor $g(x) = \sin x$, $\dfrac{d\,g(x)}{dx} = \dfrac{d \sin x}{dx} = \cos x$

5. $\dfrac{d\,(x \ln x)}{dx} = \ln x + 1$ (using the product rule)

f-line: $f(x) = x$, $\dfrac{df(x)}{dx} = \dfrac{dx}{dx} = 1$

g-line: $g(x) = \ln x$, $\dfrac{dg(x)}{dx} = \dfrac{d \ln x}{dx} = \dfrac{1}{x}$

6. $\dfrac{d\,(\dfrac{\sin x}{x})}{dx} = \dfrac{x \cos x - \sin x}{x^2}$ (using the quotient rule)

f-line: Numerator $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d \sin x}{dx} = \cos x$

g-line: Denominator $g(x) = x$, $\dfrac{d\,g(x)}{dx} = \dfrac{dx}{dx} = 1$

7. $\dfrac{d\left(\dfrac{\ln x}{x}\right)}{dx} = \dfrac{x \cdot \dfrac{1}{x} - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$

Differentiation variable? x

f(x) ? $\dfrac{\ln x}{x}$

Last operation in this function ? Division

Apply which rule? quotient rule

---

f-line: Numerator $f(x)=\ln x,\ \dfrac{d\,f(x)}{dx} = \dfrac{d\ \ln x}{dx} = \dfrac{1}{x}$

g-line: Denominator $g(x)=x,\ \dfrac{d\,g(x)}{dx} = \dfrac{dx}{dx} = 1$

8. $\dfrac{d\ \sin\ 2x}{dx} = 2\cos 2x$

Differentiation variable? x

f(x)? $\sin 2x$

Last operation in this function? $\sin(\ )$

Its argument? $2\ x$

Apply which rule? The chain rule

---

f-line: Last operation $f()=\sin(\ ),$

$\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin(g)}{dg} = \cos g = \cos 2x$

g-line: Its argument $g(x)=2x,\ \dfrac{d\,g(x)}{dx} = \dfrac{d\ 2x}{dx} = 2$

9. $\dfrac{d\ \ln\ 3x}{dx} = 3\dfrac{1}{3x} = \dfrac{1}{x}$

Differentiation variable – x
Function to differentiate – $\ln 3x$
Last operation in this function – $\ln(\ )$
Its argument – $3\,x$
Apply the chain rule

f-line: Last operation $f(\ ) = \ln(\ ),\ \dfrac{d\,f(x)}{dx} = \dfrac{d\ \ln(g)}{dg} = \dfrac{1}{g} = \dfrac{1}{3x}$

g-line: Its argument $g(x) = 3x,\ \dfrac{d\ g(x)}{dx} = \dfrac{d\ 3x}{dx} = 3$

15.3 Decision Tree for Differentiation

Generally, to differentiate a function we use the **Decision Tree for Differentiation** (see figure 15.1 below) that allows us to decide whether to use the Differentiation Table (containing derivatives of elementary functions) straight away or first use Differentiation Rules (on how to differentiate combinations of elementary functions). Once the rule is applied, even if the problem is not solved, it is reduced to one or several simpler problems which have to be treated using the Decision Tree again.

To use the Decision Tree start at the top and follow the arrows which are associated with the correct answers (if any). Use it as a formula, substituting your differentiation variable for x and your function for $f(x)$.

15.4 The higher order derivatives

If the derivative of $f(x)$ is a continuous function that has its own derivative the latter is called **the second derivative** of $f(x)$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

If the second derivative is continuous and differentiable we can find **the third derivative** of $f(x)$,

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \quad \textit{etc.}$$

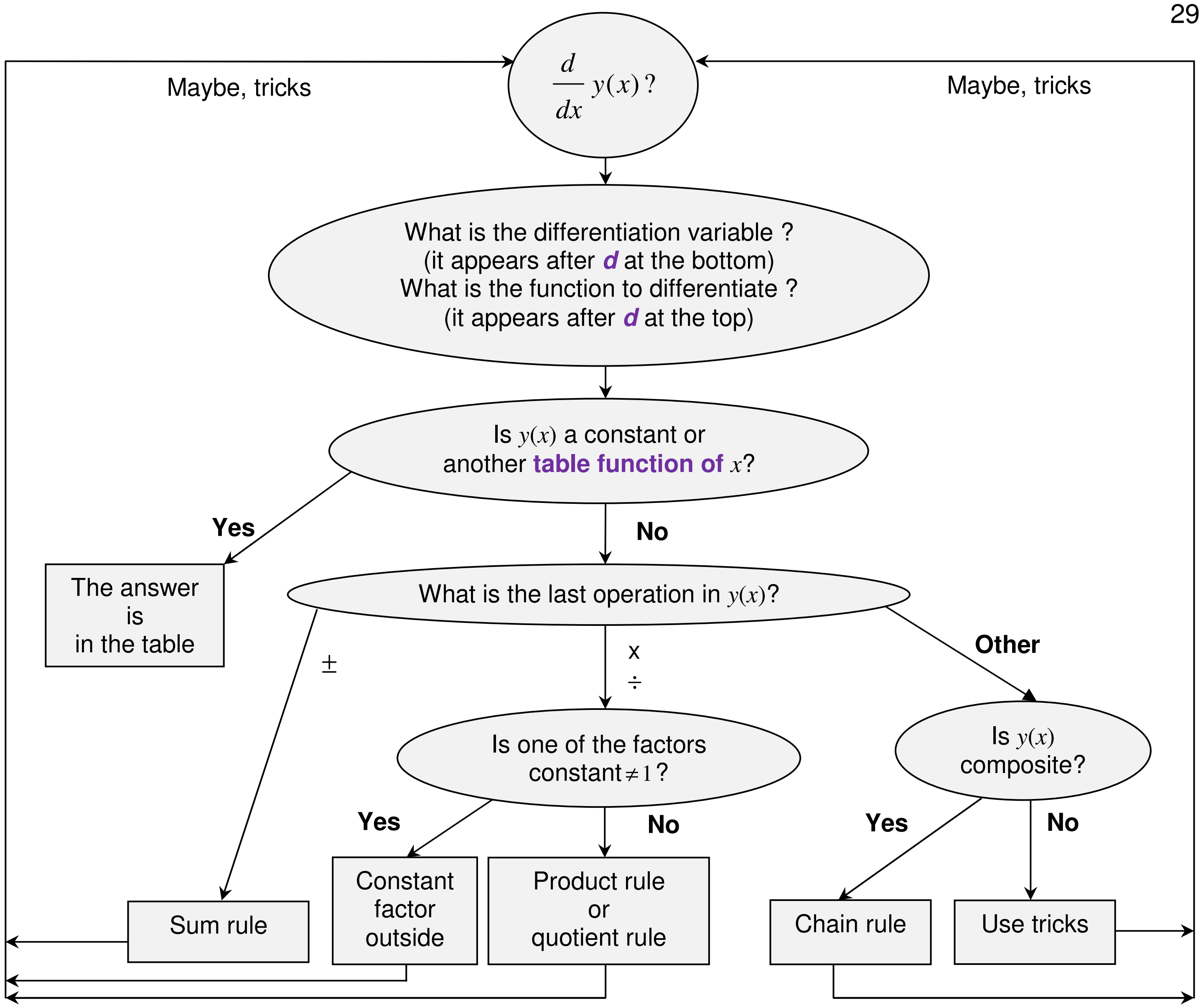


Figure 15.1. Decision Tree for Differentiation.

15.5 The partial derivatives

If we consider a function of say, two variables (two arguments) $f(x,y)$ its **partial derivatives** with respect to x and y are denoted by $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$, respectively.

15.6 Applications of differentiation

Later you will find that all simple scientific phenomena and engineering systems that you are going to study are described by differential equations that relate values of measured variables to their first or second derivatives. You will be also shown that derivatives and limits are helpful in sketching functions and thus illustrating or even predicting behaviour of the above measurable variables. Finally you will find out how derivatives can help to approximate functions, e.g. how your calculators calculate $\sin$, $\cos$ values *etc.*

15.7 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 4 and Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Lecture 13 and study Solutions to Exercises in Lecture 13 using the STUDY SKILLS Appendix**
- **Revise Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 15 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{t^2+1}}$

b) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{x^2+1}}$

Q2. Find the derivatives

a) $\dfrac{d}{du}(u^2+1)\sin u$

b) $\dfrac{d}{du}\dfrac{u^2+1}{\sin u}$

Q3. Find the derivatives

a) $\dfrac{d\cos\sqrt{x^2+1}}{dx}$

b) $\dfrac{d\cos\sqrt{x^2+1}}{dy}$

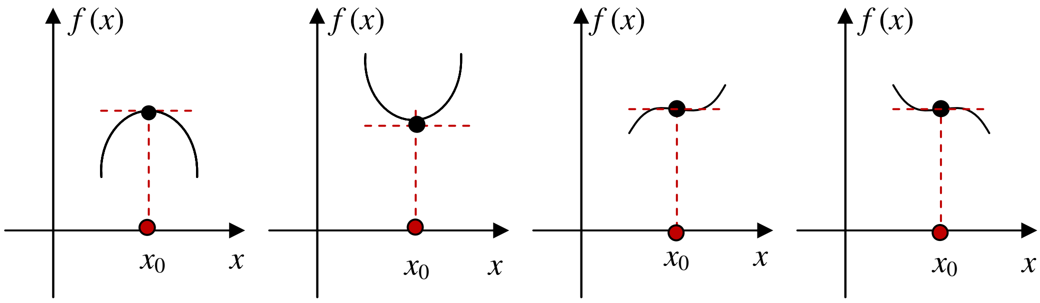
Q4. Differentiate y with respect to t if $y = (t+\sqrt{t^2+a^2})^n$.

Lecture 16. DIFFERENTIAL CALCULUS: Sketching Graphs Using Analysis

In general, any composite function can be sketched using our knowledge of limits and derivatives. In this Lecture we show how.

16.1 Stationary points

A **stationary point** is a point x_0 in the function domain where its derivative $f'(x_0) = 0$. There are four possible behaviours in the vicinity of the stationary point:



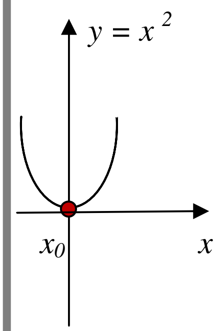
Examples: Sketch the following functions and find their stationary ponts x_0 :

1. $y = x^2$

$\frac{dy}{dx} = 0$

$2x = 0$

$x_0 = 0$

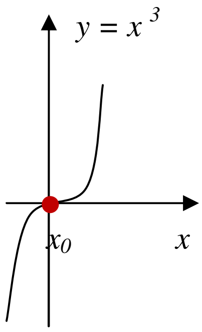


2. $y = x^3$

$\frac{dy}{dx} = 0$

$3x^2 = 0$

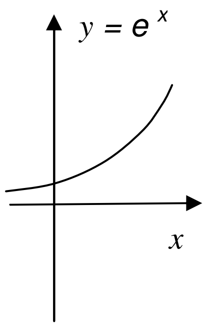
$x_0 = 0$



3. $y = e^x$

$\frac{dy}{dx} = e^x \neq 0$

No stationary points

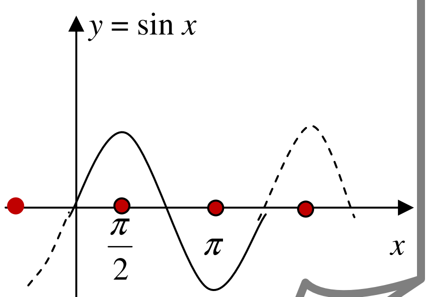


4. $y = \sin x$

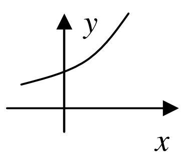
$\frac{dy}{dx} = \cos x = 0$

$x_0 = \frac{\pi}{2} n,$

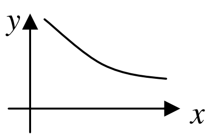
n – odd integer
($n = 2m + 1$, m – integer)



16.2 Increasing and decreasing functions



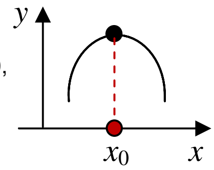
If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **increasing** on I.



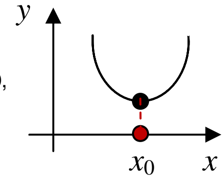
If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **decreasing** on I.

Examples: Functions $y = e^x$ and $y = \ln x$ are increasing everywhere; functions $y = |x|$ and $y = x^2$ are decreasing for $x < 0$ and increasing for $x > 0$.

16.3 Maxima and minima



If derivative (local slope) $f\,'(x) > 0$ to the left of x_0 and < 0 to the right of x_0, then function $f(x_0)$ is said to have a **maximum** at x_0.



If derivative (local slope) $f\,'(x) < 0$ to the left of x_0 and > 0 to the right of x_0, then function $f(x_0)$ is said to have a **minimum** at x_0.

Examples: $y = x^2$ has minimum at 0, x^3 has no minimum or maximum, $\sin x$ has maxima at $x = \frac{\pi}{2}m$ when m is odd and minima when m is even.

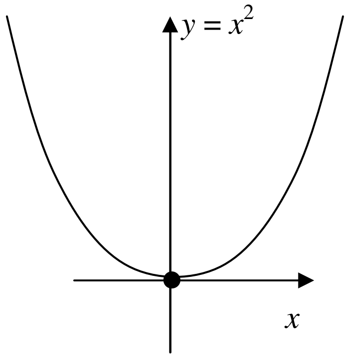
Examples:

1. Sketch $y = x^2 + 3x + 1$ by completing the square

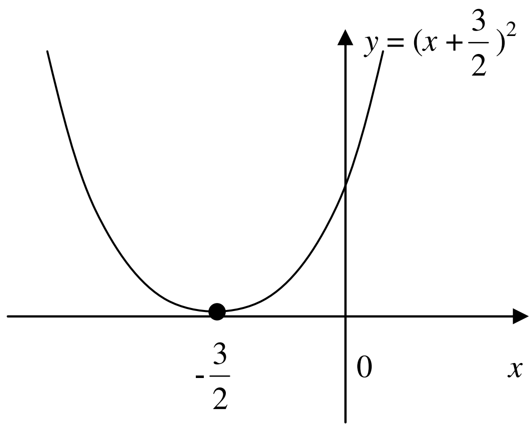
Solution:

Completing the square, $y = (x + \frac{3}{2})^2 - \frac{9}{4} + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$.

Step 1. Using the recipe for sketching by simple transformations, we first drop all constant factors and terms and sketch the basic shape of the function that remains:

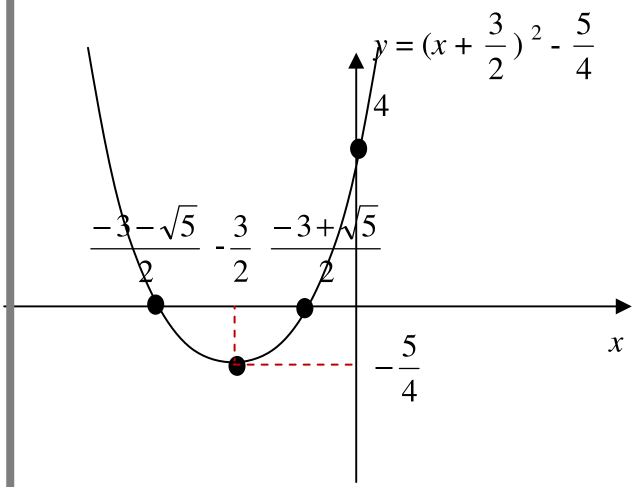


Step 2. We then reintroduce the constant factors and terms that appeared in the original functional equation one by one, in Order of Operations and sketch the resulting functions underneath one another:



$+\frac{3}{2}$ – constant term, first operation

$\Rightarrow$ translation wrt x-axis by $-\frac{3}{2}$



$-\frac{5}{4}$ – constant term, first operation

$\Rightarrow$ translation wrt y-axis by $-\frac{5}{4}$

Step 3.

Intersection with the y-axis: $x = 0,\ \ y = 4$

Intersection with the x-axis: $y = 0,\ \ x_{1,2} = -\dfrac{3 \pm \sqrt{9-4}}{2} = -\dfrac{3 \pm \sqrt{5}}{2} \approx -\dfrac{3 \pm 2.1}{2} \approx -2\dfrac{1}{2}, -\dfrac{1}{2}$

2. Sketch $y = x^2 + 3x + 1$ by analysis.

Solution

Step 1. **Intersection with the y-axis:** $x = 0,\ \ y = 4$

Intersection with the x-axis: $y = 0,\ \ x_{1,2} = -\dfrac{3 \pm \sqrt{9-4}}{2} = -\dfrac{3 \pm \sqrt{5}}{2} \approx -\dfrac{3 \pm 2.1}{2}$

Step 2. Find and sketch the first derivative

$$y' = 2x + 3$$

a) Find where the first derivative is zero: $2x + 3 = 0 \Rightarrow\ x = -\dfrac{3}{2}$

$\Rightarrow$ Stationary point of the function is $x = -\dfrac{3}{2}$

b) Find where the first derivative is positive and where it is negative:

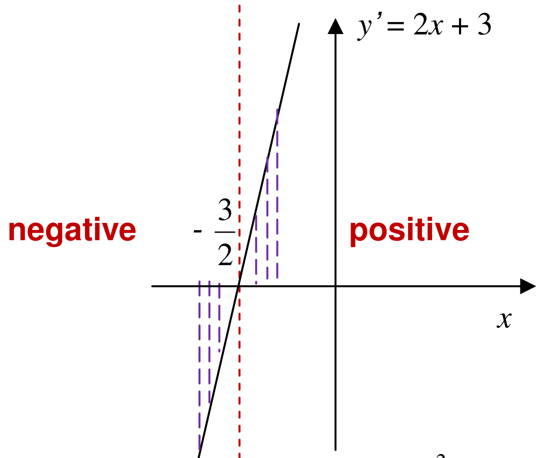
Can do so by solving the inequalities

$2x + 3 > 0\ \ (y' > 0)$ for $x > -\dfrac{3}{2}$

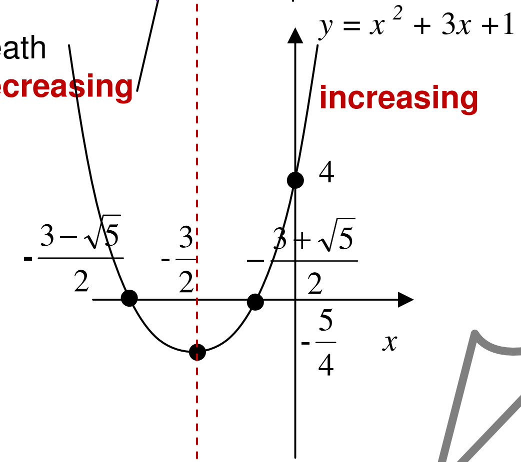
$2x + 3 < 0\ \ (y' < 0)$ for $x < -\dfrac{3}{2}$

Note: inequalities can be solved the same way as equations but when multiplying by negative factors the inequality sign should be reversed.

or graphically



Step 3: Sketch the function y directly underneath the first derivative y'



Step 4: Find the minimum function value

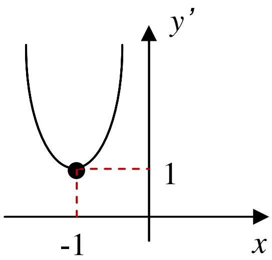
$$y\left(-\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 1 = \frac{9-18+4}{4} = -\frac{5}{4}$$

3. Sketch $y = x^3 + 3x^2 + 4x + 1$ by analysis

Solution

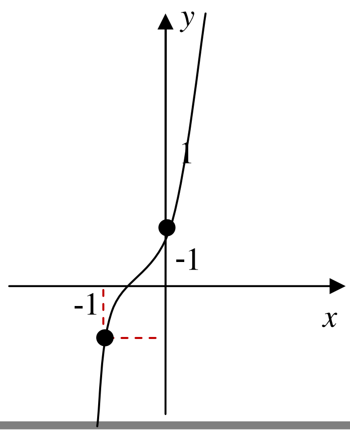
Step 1. Find and sketch the first derivative y'

$$y' = 3x^2 + 6x + 4 = 3\,(x^2 + 2x) + 4 = 3\,[(x+1)^2 - 1] + 4 = 3\,(x+1)^2 + 1 \;>\; 0$$



$y' > 0$ everywhere

Step 2. Sketch the function y underneath



y is increasing everywhere

Special points: $y(0) = 1$

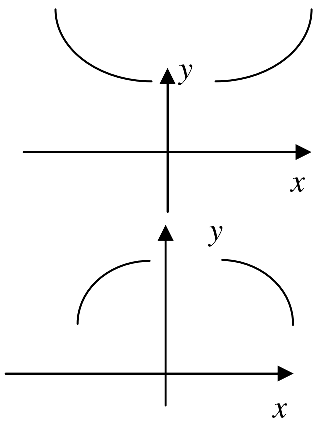
$y(-1) = -1$ (not that special – but easy to find)

Optional

16.5 Convex and concave functions

If the second derivative $f''(x) > 0$ on interval I the first derivative $f'(x)$ is increasing on I, and function $f(x)$ is said to be **convex** on interval I.

If the second derivative $f''(x) < 0$ on interval I the first derivative $f'(x)$ is decreasing on I and function $f(x)$ is said to be **concave** on interval I.



Examples: establish the regions of convexity of the following elementary functions:

$y = x^2$

$y' = 2x$

$y'' = 2 > 0$

$\Rightarrow y = x^2$ is convex everywhere

$y = x^3$

$y' = 3x^2$

$$y'' = 6x \begin{cases} > 0 \text{ if } x > 0 \\ < 0 \text{ if } x < 0 \end{cases}$$

$$\Rightarrow y = x^3 \text{ is } \begin{cases} \text{convex if } x > 0 \\ \text{concave if } x < 0 \end{cases}$$

$y = e^x$

$y' = e^x$

$y'' = e^x > 0$

$\Rightarrow y = e^x$ is convex everywhere

16.6 Inflexion points

If the second derivative $f''(x_0) = 0$ and $f''(x)$ changes sign at the stationary point x_0, then the stationary point x_0 is called **an inflexion point.**

Example: $y = x^3$ has an inflexion point at $x = 0$.

Thus, there is another way of determining whether the stationary point is at a maximum or minimum:

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the function is convex in the vicinity of x_0 and x_0 **is at a minimum.**

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the function is concave in the vicinity of x_0 and x_0 **is at a maximum.**

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then ***further*** **checks are required.**

16.7 Sketching rational functions using analysis

We illustrate the general principles of sketching by analysis by applying the method to rational functions.

A **rational function** $R(x)$ is

$$R(x) = \frac{P(x)}{Q(x)},$$

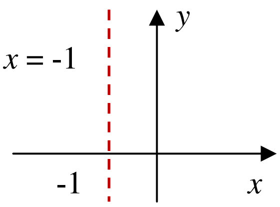
where P and Q are polynomials.

Examples:

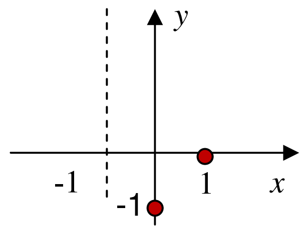
1. Sketch the rational function $y = \dfrac{x-1}{x+1}$

Solution

Step 1. Domain: $x \neq -1$ ($x = -1$ is called **a pole**, the dashed line $x = -1$ is called **a vertical asymptote**)

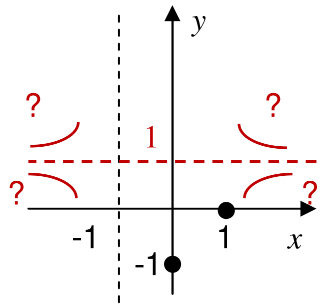


Step 2. Intersection with the y-axis: $x = 0$, $y = -1$
Intersection with the x-axis: $y = 0$, $x = 1$



Step 3. Behaviour at the domain boundaries:

$x \to \infty$, $y \to 1$
$x \to -\infty$, $y \to 1$

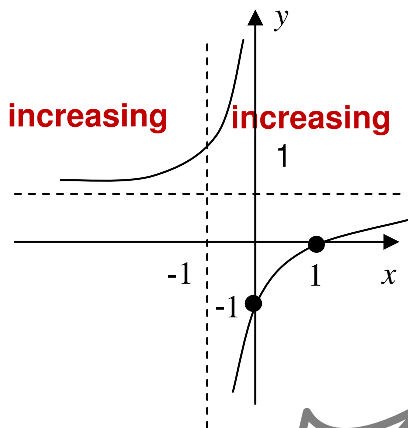


The dashed line $y=1$ is called **a horizontal asymptote**.
We know that the function approaches it when $|x|$ grows large but we do not know how.

Step 4. Find the first derivative:

$$\frac{dy}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$$

$\Rightarrow$ y is increasing everywhere



2. Sketch $y = \dfrac{x-1}{x^2-4x+4}$

Solution

Step 1. Domain: $x \neq 2$ (x=2 is a pole, the line is a vertical asymptote)

Step 2. Intersection with the y-axis: $x = 0,\ \ y = -\dfrac{1}{4}$

Intersection with the x-axis: $y = 0,\ \ x = 1$

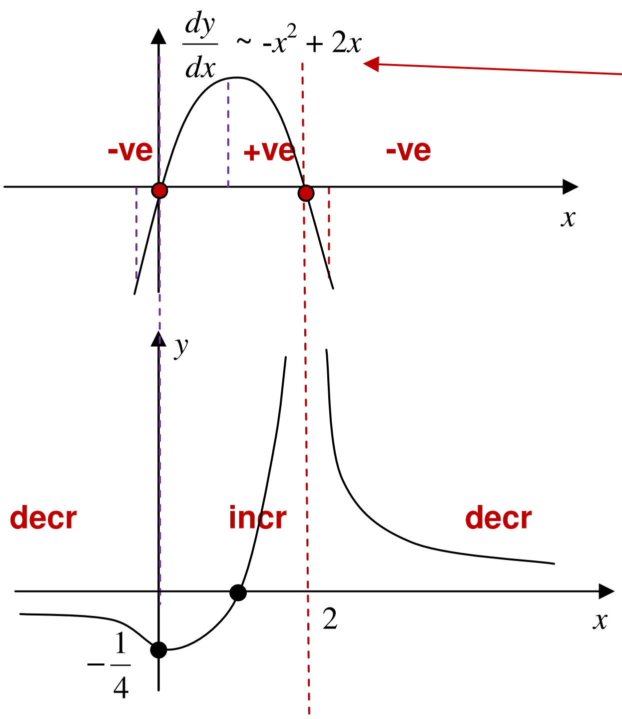
Step 3. Behaviour at the domain boundaries

$x \longrightarrow \pm\infty$, $y \longrightarrow \pm 0$

The line $y = 0$ is a **horizontal asymptote**

Step 4. Find the first derivative:

$$\frac{dy}{dx} = -\frac{x}{(x-2)^3} = \frac{-x^2+2x}{(x-2)^4}\begin{cases} = 0 & \text{at } x = 0 \\ > 0 \ (y \text{ is increasing}) & \text{for } 0 < x < 2 \\ < 0 \ (y \text{ is decreasing}) & \text{for } x < 0 \ \& \ x > 2 \end{cases}$$



This is not a derivative but it is 0, positive and negative where $\dfrac{dy}{dx}$ is – since $(x-2)^4$>0 when $x \neq 2$.

The sign ~ means "behaves as".

Note that $x = 2$ turns the numerator $-x^2+2x$ into zero but $\dfrac{dy}{dx}$ is not zero at this point. The "phantom" zero appears because we multiplied both numerator and denominator of $\dfrac{dy}{dx}$ by $(x - 2)$. This was a trick performed to make sure that the denominator is never negative.

Optional

polynomial degree 2

Sketch y = $\dfrac{x^2+4x+2}{x+1}$ - **improper rational function**

polynomial degree 1

Difference in degrees = 1 $\Rightarrow$ the whole part is a **polynomial degree** 1

$$\frac{x^2+4x+2}{x+1} = Ax+B + \frac{C}{x+1}$$

polynomial degree 0

$\Uparrow$

polynomial degree 1

To find A, B and C use

1. **long division** or
2. **partial fractions method**

$$\frac{x^2+4x+2}{x+1} = Ax + B + \frac{C}{x+1} \Big/ (x+1)$$

$$x^2 + 4x + 2 = Ax(x+1) + B(x+1) + C$$

True for all x $\Rightarrow$ can choose any convenient values of x

1. Choose values that "kill" terms $x = -1$: $1 - 4 + 2 = C$, $C = -1$
   $x = 0$: $2 = B + C$, $2 = B - 1$, $B = 3$
2. Equate coefficients of the highest power: $1 = A$

$$y = x + 3 - \frac{1}{x+1}$$

Now we have to sketch $y = x + 3 - \dfrac{1}{x+1}$

Step 1. Domain: $x \neq -1$ ($x = -1$ is a pole)

Step 2. Intersection with the y-axis: $x = 0$, $y = 2$

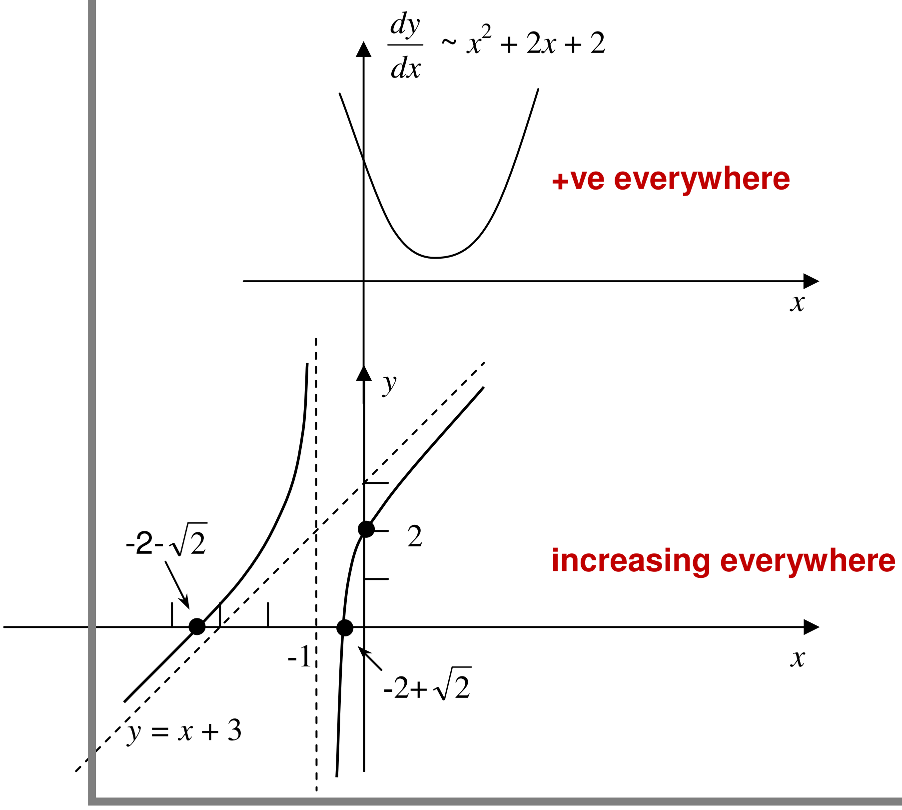
Intersection with the x-axis: $y = 0$, $x^2 + 4x + 2 = 0$, $x_{1,2} = -2 \pm \sqrt{2}$

Step 3. Behaviour at the domain boundaries:

$x \to \pm\infty$, $y \to x + 3$ – **an oblique asymptote** (typical for improper rational fractions)

Step 4. Find the first derivative:

$$\frac{dy}{dx} = \frac{x^2+2x+2}{(x+1)^2} > 0 \text{ because } x^2 + 2x + 2 = 0 \text{ at } x_{1,2} = -1 \pm j \text{ (no real roots)}$$



16.8 Applications of rational functions and their graphs

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. Any such transfer function is a rational function of one complex variable. Sketching magnitudes and phases of these functions helps design efficient control systems.

16.9 Instructions for self-study

- **Revise Lecture 9 and study Solutions to Exercises in Lecture 9 (sketching by simple transformations)**
- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lectures 13 - 15 (limits and differentiation) and study Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 16 using the STUDY SKILLS Summary**
- **Do the following exercises:**

Q1. Sketch using analysis

a) $y = -3\,(x^3 - 3x^2 + x)$

b) $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Q2. Sketch $y = \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q3. Sketch $y = u(x+2)\ \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q4. Sketch $y = \dfrac{x^2 - 5x + 6}{x + 1}$

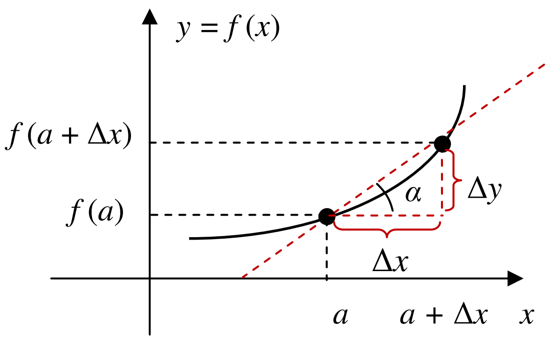
Lecture 17. Application of DIFFERENTIAL CALCULUS to Approximation of Functions: the Taylor and Maclaurin Series

17.1 Approximating a real function of a real variable using its first derivative

Let us discuss how we can **approximate** the value of function $f(x)$ at $a+ \Delta x$ if we know its value at a. It is clear from the figure below that

$$f(a+\Delta x) \approx f(a) + \Delta y\,.$$

How can we approximate Δy? To see this, let us revise the definition of the first derivative given in Lecture 13.



Question: What is the geometrical interpretation of derivative?
Answer: It is a local slope, $f'(x) = \tan\alpha$
Question: How can we find its approximate value?
Answer: $\tan\alpha \approx \dfrac{\Delta y}{\Delta x}$ when Δx is small.
Question: How to solve this approximate equality for Δy?
Answer: Multiplying both sides by Δx we can see that $\Delta y \approx \tan\alpha\,\Delta x = f'(x)\Delta x$

$$\Rightarrow f(a+\Delta x) \approx f(a) + f''(a)\,\Delta x$$

Verbalise: The function value near x **approximately equals** the function value at x + derivative at x times distance to x.

This is an approximation linear in Δx (in the vicinity of x the graph of $f(x)$ looks like a straight line). Sometimes we need approximations based on quadratic, cubic … *n-th* degree polynomials. Before showing how we introduce another way to represent a polynomial.

17.2 The Maclaurin polynomials

Question: What is a polynomial?
Answer:

Question: Conventionally we write the first term of the polynomial $P_n(x)$ as $a_n x^n$. All coefficients are constant with respect to x **and are denoted by letter** a. What will be the next term and the next and the next…?
Answer:

Any polynomial can be re-written in another form as

the Maclaurin polynomial: $P_n(x) = P_n(0) + P_n'(0)x + \dfrac{P_n''(0)}{2!}x^2 + \dfrac{P_n'''(0)}{3!}x^3 + \ldots \dfrac{P_n^{(n)}(0)}{n!}x^n$

where n **factorial** $n! = 1\times2\times3\times4\times\ldots\times n$.

We can show this using the following steps:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 \quad \Rightarrow \quad P_n\ (0) = a_0 \quad \Rightarrow \quad a_0 = P_n\ (0)$$

$$P_n'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + 3a_3 x^2 + 2a_2 x \ + a_1 \quad \Rightarrow \quad P_n\ (0) = a_1 \quad \Rightarrow \quad a_1 = P_n'\ (0)$$

$$P_n''(x) = a_n n(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 x \ + 2a_2 \Rightarrow P_n''(0) = 2a_2 \Rightarrow a_2 = \frac{P_n''(0)}{2}$$

$$P_n'''(x) = a_n n(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 \quad \Rightarrow \quad P_n'''(0) = \ 2\cdot 3\ a_3 \Rightarrow a_3 = \frac{P_n'''(0)}{3!}$$

… …

$$P_n^{(n)}(x) = a_n n(n-1)(n-2)....[n-(n-1)]x^{n-n} \quad \Rightarrow \quad P^{(n)}(0) = n!\ a_n \quad \Rightarrow \quad a_n = \frac{P^{(n)}(0)}{n!}$$

$$\Rightarrow P_n(x) = P_n(0) + P_n'(0)x + \frac{P_n''(0)}{2!}x^2 + \frac{P'''(0)}{3!}x^3 + ... \frac{P_n^{(n)}(0)}{n!}x^n$$

Verbalise: The Maclaurin polynomial at x is the polynomial at 0 + 1st derivative at 0 times x + 2nd derivative at 0 times x squared over $2!$ + ….

17.3 The Taylor polynomials

In the same way we can prove that any polynomial can be re-written as

the Taylor polynomial: $P_n(x) = P_n(a) + P_n'(a)(x-a) + ... \frac{P_n^{(n)}(a)}{n!}(x-a)^n$

Verbalise: The Taylor polynomial at x is the polynomial at a + derivative at a times the signed distance from x to a + ….

Note: in the Maclaurin series x can be viewed as the signed distance from x to 0.

17.4 The Taylor series

Many differentiable functions $f(x)$ can be represented using the infinite

Taylor series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-2)^2 + ...$

Verbalise: The function value at x is the function value at a + derivative at a times the distance to a + ….

These functions can be **approximated** as polynomials using **truncated Taylor series** (that is, Taylor polynomials).

Consider the approximation error **Optional**

$$e_n(a, x) \equiv f(x) - P_n(a, x)$$

If the error $e_n(x) \to 0$ as $n \to \infty$, then $f(x)$ equals its Taylor series at x and $y(x) \approx P_n(a, x)$ is an approximation based on the n-th order Taylor polynomial.

For e^x, $\sin x$, $\cos x$, $e_n(x - a) \to 0$ faster than $(x - a)^n$

Proving the above results and checking the limiting behaviour of the error e_n is rather involved.

17.5 The Maclaurin series

If $a = 0$, the Taylor series is called the **Maclaurin series.**

Examples:

1. Find the Maclaurin series for e^x.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$ $\ldots$ $f^{(n)}(x) = e^x$	$f'(0) = 1$ $f''(0) = 1$ $f^{(n)}(0) = 1$

Step 4. Substitute the above values into the general the Maclarin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \ \ldots$$

$\Rightarrow e^x \approx 1 + x, |x| < 1$ (in the vicinity of 0 the graph of e^x is almost a straight line)

2. Find the Maclaurin series for $\sin x$.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f'(x) = \cos x$ $f''(x) = -\sin x$ $f'''(x) = -\cos x$ $f^{(IV)}(x) = \sin x$ $\ldots$	$f'(0)=1$ $f''(0)=0$ $f^{(IV)}(0)=0$

Step 4. Substitute the above values into the general Maclarin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$\Rightarrow \sin x \approx x, |x| < 1$ (in the vicinity of 0 the graph of $\sin x$ is almost a straight line)

c) Find the Maclaurin series for $\cos x$.

Solution:

Step 1. $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \ldots$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = \cos x$ $f'(x) = -\sin x$ $f''(x) = -\cos x$	$f(0) = 1$ $f'(0) = 0$ $f''(0) = -1$

Step 4.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \quad -$$

$\Rightarrow \cos x \approx 1 - \dfrac{x^2}{2}, \; |x| < 1$ (in the vicinity of 0 the graph of $\cos x$ looks like a parabola – reflected **wrt** (with respect to) the horizontal axis shifted up by 1).

Thus, while any polynomial can be **re-written** as a Taylor or MacLaurin polynomial, many other n time differentiable functions can be **approximated** by Taylor or MacLaurin polynomials $P_n(x)$. The discussion on whether such approximation is possible or how to decide on the optimal degree of the approximating polynomial lies outside the scope of these notes.

17.6 L'Hospital's rule

Let us find $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ when $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$, so that we have indeterminacy $\dfrac{0}{0}$.

In this case we can write

:$(x - a)$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)(x-a)+\ldots}{g'(a)(x-a)+\ldots}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

The recipe

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\ \text{ if }\ \lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$$

is called **L'Hospital's** rule.

If still get the $\dfrac{0}{0}$ indeterminacy – continue applying the rule (find the second derivatives of the numerator and denominator, their third derivatives *etc.*)

Examples:

1. Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution: $\lim\limits_{x\to 0} x=\lim\limits_{x\to 0}\sin x=0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1$

2. Find $\lim\limits_{x\to 0}\dfrac{1-e^x}{x}$.

Solution: $\lim\limits_{x\to 0} x=\lim\limits_{x\to 0} 1-e^x=0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0}\dfrac{1-e^x}{x}=\lim\limits_{x\to 0}\dfrac{-e^x}{1}=-1$

17.7 Applications

The above series are used by calculators to give approximate values of elementary functions, $\sin$, $\cos$, exponent *etc.*

17.8 Instructions for self-study

- **Revise ALGEBRA Summary (smile rule)**
- **Revise Lecture 5 (polynomials)**
- **Revise FUNCTIONS Summary**
- **Revise Lectures 13 - 14 (limits)**
- **Revise Lectures 14 - 15 (differentiation) and Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**

- **Revise Lecture 16 using the STUDY SKILLS Appendix**
- **Study Lecture 17 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find the first three non-zero terms in the Maclaurin series for $y = \frac{1}{1+x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y = \frac{1}{1+x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\frac{4}{5}$

d) Find $\lim_{x \to -1} [\frac{1}{1+x}(1 - e^{1+x})]$.

Q2.

a) Find the first three non-zero terms in the Maclaurin series for $y = \frac{1}{1-x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y = \frac{1}{1-x}$ in the vicinity of $x = 0$.

c) Find, without using a calculator, the approximate value of $\frac{4}{3}$

d) Find $\lim_{x \to 1} [\frac{1}{1-x} \sin(1-x)]$

Q3.

a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1+x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y = \sqrt{1+x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{3}}$

d) Find $\lim_{x \to -1} \frac{\sqrt{1+x}}{1-x^2}$

Q4.

a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1-x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y = \sqrt{1-x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\sqrt{\frac{4}{5}}$

d) Find $\lim_{x \to 1} \frac{\sqrt{1-x}}{1-x^2}$

IV. SUMMARIES

Algebra Summary

OPERATIONS

Addition (direct operation)

Addition of whole numbers gives whole number

1. $a + b = b + a$

Terminology: a and b are called **terms**
$a + b$ is called **sum**

2. $(a + b) + c = a + (b + c)$

Subtraction (inverse operation)

Def : $a - b = x$**:** $x + b = a$

Note: $a + b - b = a$ **(subtraction undoes addition)**
$a - b + b = a$ **(addition undoes subtraction)**

3. $a + 0 = a$

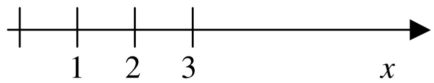
4. for each $\underline{a}$ there exists one **additive inverse** $-a$:
$a + (-a) = 0$

Rules (follow form Laws):

$+ (b + c) = + b + c$
$+ \ a + b \ = \ a + b$
$- (- a) = a$
$- (a) = -a$

TYPES OF VARIABLES

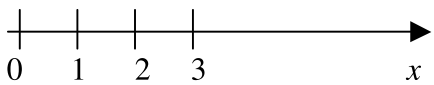
Whole numbers are $1, 2, 3, ...$



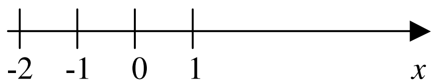
introduces **0** and **negative numbers**:

$a - a = 0$
if $b > a$ $\quad a - b = - (b - a)$

Natural numbers are $0, 1, 2, ...$



Integers are ..., -2, -1, 0, 1, 2, ...



Multiplication (direct operation)

For whole numbers n

$$a \cdot n = \underbrace{a + \ldots\ldots\ldots\ldots + a}_{n \text{ times}}$$

Notation: $ab = a \cdot b = a \times b$

$2b = 2 \cdot b = 2 \times b$

$23 \neq 2 \cdot 3,\ 23 = 2 \cdot 10 + 3$

$2\frac{1}{2} \neq 2 \cdot \frac{1}{2},\ 2\frac{1}{2} = 2 + \frac{1}{2}$

$2\frac{3}{2} = 2 \cdot \frac{3}{2}$

1. $a \cdot b = a \cdot b$

Terminology: a and b are called **factors**

ab - **product**

2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Conventions: $abc = (ab)c$

$a(-bc) = -abc$

3. $a(b+c) = ab+ac$

$\longrightarrow$ Removing brackets

$\longleftarrow$ Factoring

4. $a \cdot 0 = 0$
5. $a \cdot 1 = a$

Rules (follow from Laws):

$(a + b)(c + d) = ac + ad + bc + bd$ (**SMILE RULE**)

$(-1) \cdot n = -n$

$(-1) \cdot (-1) = 1$

Division (inverse operation)

Def: $a/b = x$: $xb = a$

Terminology: a – **numerator**

b – **denominator**

a/b – **fraction (ratio)**

proper fraction if $|a|<|b|$, a, b integers

Note: $ab/b = a$ **(division undoes multiplication)**

$(a/b)b = a$ **(multiplication undoes division)**

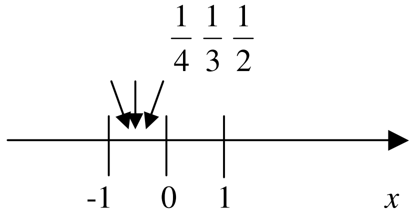
6. For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $a \cdot 1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\frac{m}{n}$,

where m and $n \neq 0$ are integers

(**division by zero is not defined**)



Rules:

$$\frac{a}{b}\cdot n=\frac{an}{b}$$

$$\frac{a/b}{n}=\frac{a}{bn}=\frac{a/n}{b}$$

$$\frac{an}{bn}=\frac{a\cdot\cancel{n}}{b\cdot\cancel{n}}=\frac{a}{b}$$ **CANCELLATION**

$$\frac{\frac{1}{n}}{m}=\frac{m}{n}$$ **FLIP RULE**

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

Note: $\frac{a+c}{b}=(a+c)/b$

$$\frac{a}{b}^{d}+\frac{c}{d}^{b}=\frac{ad}{bd}+\frac{cb}{db}=\frac{ad+cb}{bd}$$ **COMMON DENOMINATOR RULE**

n-th **power** b^n (direct operation)
If n – a whole number

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

Rules

$a^m \cdot a^n = a^{m+n}$

(product of powers with the same base is a power with indices added)

$a^n \cdot b^n = (ab)^n$

(product of powers is power of product)

$a^m/a^n = a^{m-n}$

(ratio of powers is power of ratios)

$a^m/b^m = (a/b)^m$

(ratio of powers with the same base – subtract indices)

$a^0 = 1$

$a^{-n} = 1/a^n$

$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

n-th **root** (inverse to taking to power n)

Def: $\sqrt[n]{b} = x$: $x^n = b$

Note:

$\sqrt[n]{b^n} = b$

(taking *n-th* root undoes taking *n-th* power)

$(\sqrt[n]{b})^n = b$

(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = \sqrt[n]{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

Logarithm base b (inverse to taking b to power)

Def: $\log_b a = n$: $b^n = a$

Note:

$\log_b b^n = n$ (check using definition: $b^n = b^n$)

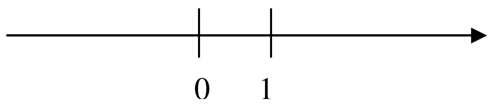
(taking $\log_b$ undoes taking b to power)

$b^{\log_b n} = n$

(**taking b to power undoes log $_b$**)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, …

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole real line



introduces **irrational** (not rational) **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers**, $\sqrt{-1}, \log_{10}(-1)$, *etc.*

Rules (follow from Rules for Indeces):

$\log_b xy = \log_b x + \log_b y$

(log of a product is sum of logs)

$\log_b x/y = \log_b x - \log_b y$

(log of a ratio is difference of logs)

$\log_b 1 = 0$ **(log of 1 is 0)**

$\log_b b = 1$ **($b^1 = b$)**

$\log_b 1/a = -\log_b a$

$\log_b x^n = n \log_b x$

(log of a power is power times log)

$\log_b a = \log_c a / \log_c b$

(changing base)

General remarks

1. $a - b = a + (-b)$ $\longrightarrow$ a difference can be re-written as a sum

2. $\frac{a}{b} = a \cdot \frac{1}{b} = ab^{-1}$ $\longrightarrow$ a ratio can be re-written as a product

3. $\sqrt[n]{b} = b^{1/n}$ $\longrightarrow$ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

6. By convention, in any algebraic expression operations should be performed using the Order of Operations convention.

Functions Summary

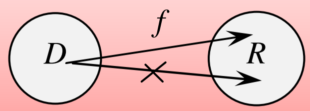
Variables are denoted mostly by $x, y, z, p, \ldots.. w$.
A variable can take any value from a set of allowed numbers.

Functions are denoted mostly by f, g and h or $f\,(\,)$, $g\,(\,)$ and $h(\,)$
(no multiplication sign is intended!).

In mathematics, the word **function** has two meanings:

1) $f()$ - an operation or a chain of operations on an **independent variable (argument)**;
2) $f\,(x)$ – a **dependent variable** (a variable dependent on x), that is the result of applying the operation $f\,(\,)$ to an independent variable x.

A diagrammatical representation of a function



To specify a function we need to specify a (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, $f\,(x)$ assigns one and only one value $y \in R$ (**range**, the set of all possible values of the dependent variable).

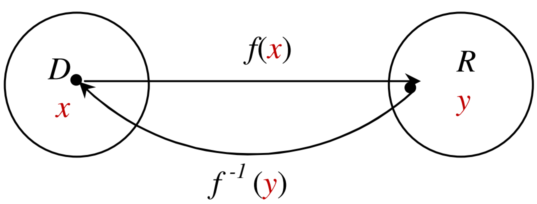
Inverse functions

$f^{-1}(x)$: $f \circ f^{-1}(x) = f^{-1} \circ f\,(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function



Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. The **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**, and some authors do not bracket **arguments** of elementary **functions**, such as $\exp$, $\log$, $\sin$, $\cos$, $\tan$ *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.*

Other **operations** must be performed in the order of decreasing complexity, which is

oiB - operations in Brackets (including implicit)

F - Functions $f(\,)$

P - Powers (including inverse operations of roots and logs)

M - Multiplication (including inverse operation of division)

A - Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of operations (OOO)

1. Make implicit (invisible) brackets visible (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO

oiB	f( )	P	M	A
(including implicit)		$(\ )^{(\ )}$	$\times$	+
		roots	$\div$	-
		logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2 + bx + c\,,$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared): b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2 + bx + c = 0\,.$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be 360° or 2π (radians)

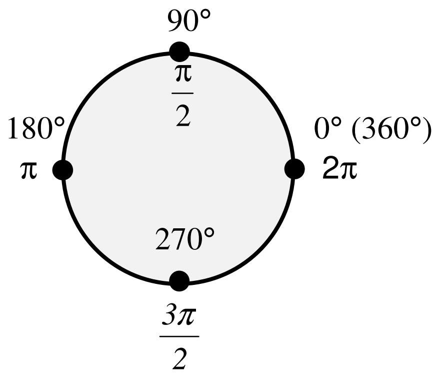
$$\Rightarrow 2\pi\ (rad) = 360°$$
$$\Rightarrow 1\ (rad) \approx 57°$$

The radian is a dimensionless unit of angle.

$$\Rightarrow \quad x\,(rad) = x\,(rad)\,\frac{180^{o}}{\pi(\text{rad})} = y^{o}\ , \quad y^{o} = y^{o}\,\frac{\pi(\text{rad})}{180^{o}} = x\,(rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

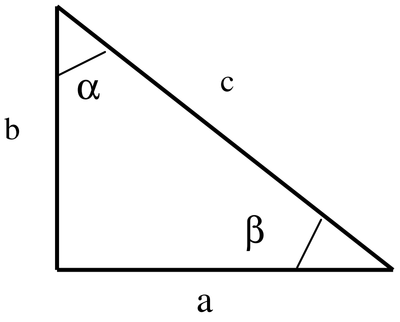
Frequently used angles



$30° = (\frac{30\pi}{180} =)\ \frac{\pi}{6}$ $60° = (\frac{60\pi}{180} =)\ \frac{\pi}{3}$ $120° = (\frac{120\pi}{180} =)\ \frac{2\pi}{3}$	$45° = \frac{\pi}{4}$

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90^0) as follows:



$$\sin\alpha = \cos\beta = \frac{a}{c}$$

$$\cos\alpha = \sin\beta = \frac{b}{c}$$

$$\tan a = \cot\beta = \frac{a}{b}$$

$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Trigonometric identities

$\sin^2 x + \cos^2 x = 1$
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$

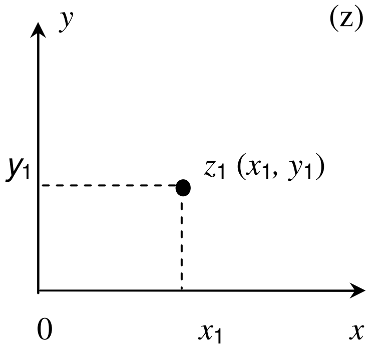
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos 2x = \cos^2 x - \sin^2 x$
$\sin 2x = 2\sin x \cos x$

$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x + \alpha)$, where $\tan\alpha = \frac{A}{B}$

Complex Numbers

The Cartesian representation of a complex number

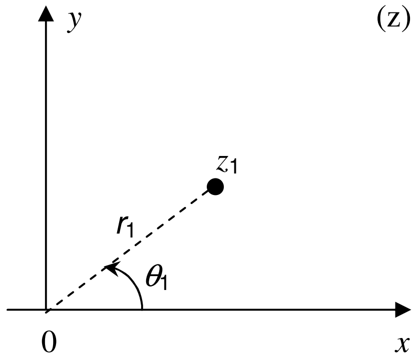


real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z = x + jy$, where x, y – real, $j = \sqrt{-1}$

x-coordinate y-coordinate (on the ordinary plane)

The exponential representation of a complex number



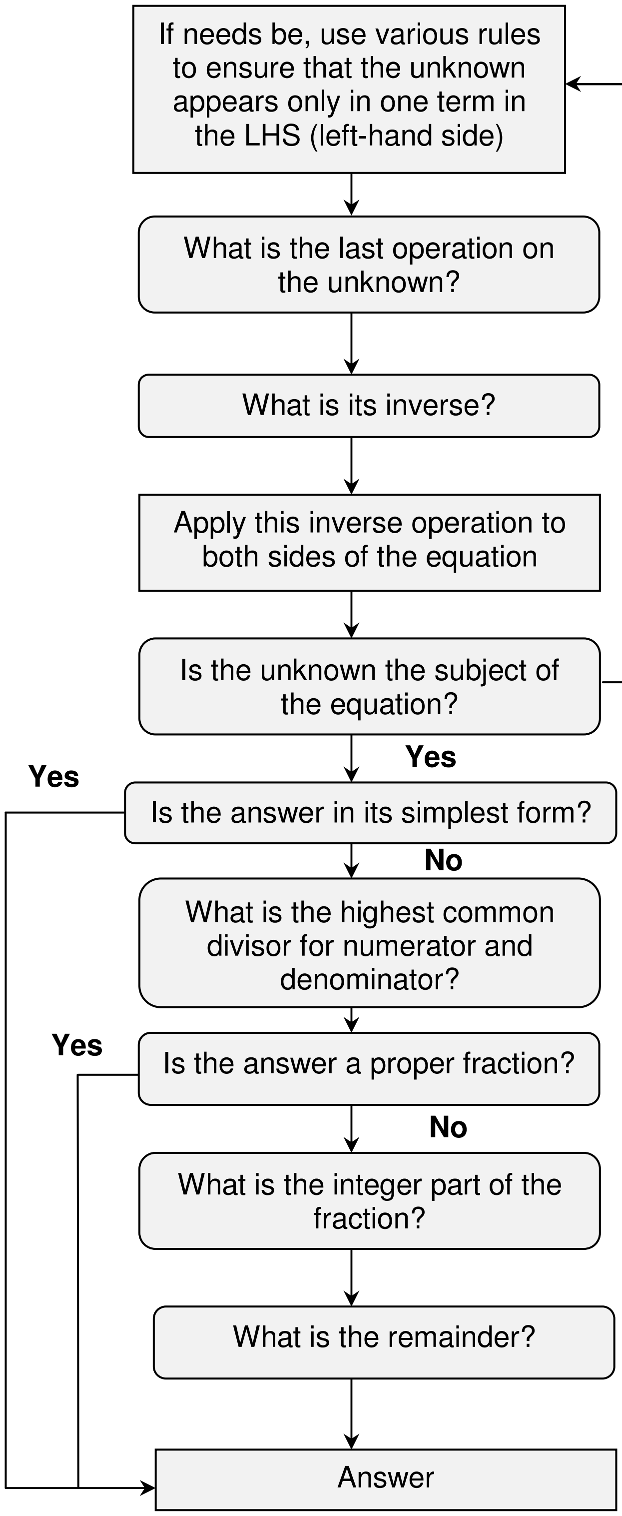
magnitude, $|z|$ **argument,** $\arg(z)$ (on the Argand diagram)

$z = r\,e^{j\theta}$, where $r \geq 0$, θ - angle
(in degrees or radians)

polar radius r
(distance to the origin)

polar angle θ (on the ordinary plane)

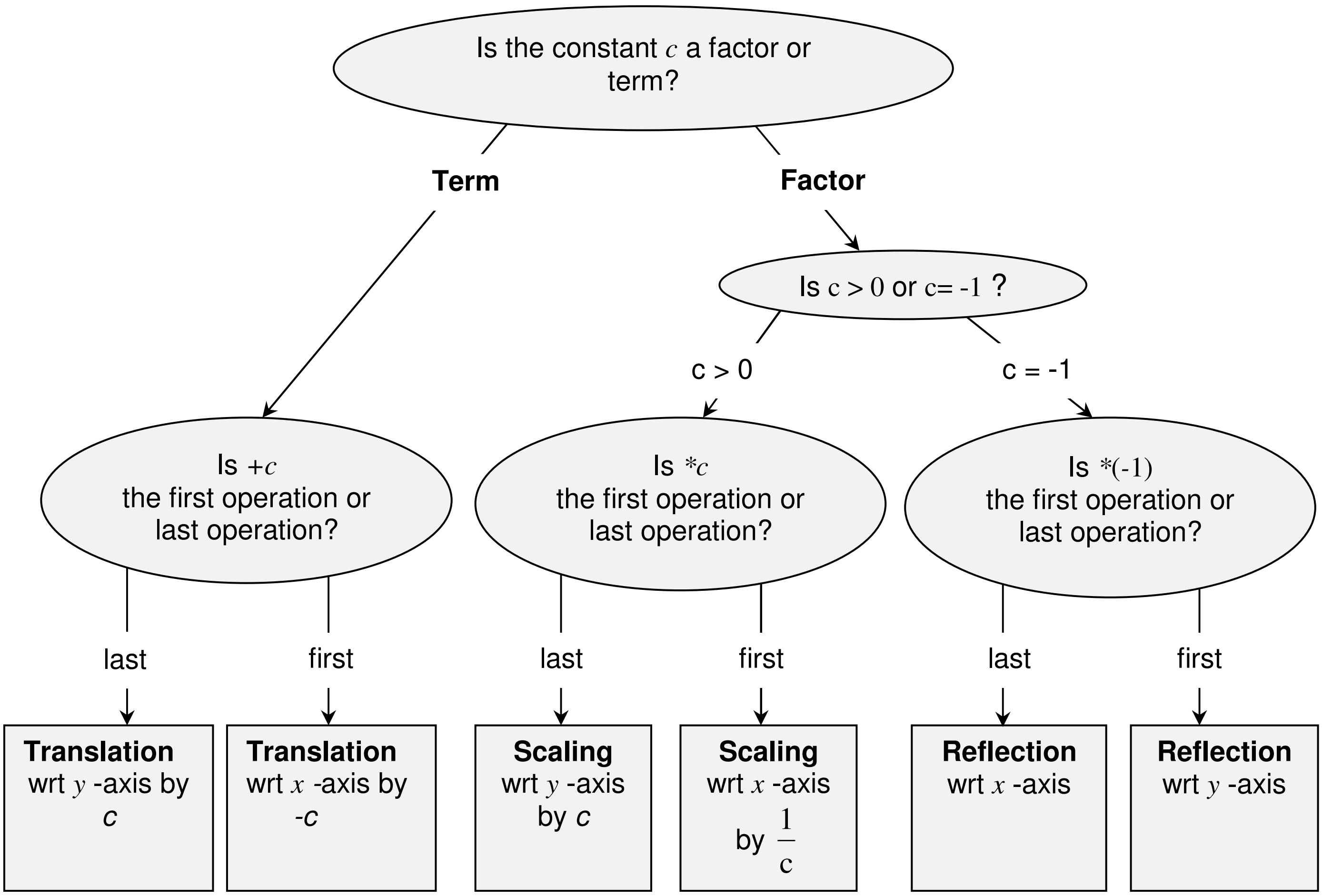
Decision Tree For Solving Simple Equations



Sketching Graphs by Simple Transformations

1. Drop all constant factors and terms.
2. Bring the constants back one by one in the Order of Operations (not necessary but advisable) and at each Step use the Decision Tree given below to decide which simple transformation is affected by each constant. Sketch the resulting graphs underneath one another.

DECISION TREE FOR SKETCHING $y=f(x)+c$, $y=f(x+c)$, $y=cf(x)$and $y=f(cx)$



Note 1: if c is a **negative factor**, write $c = (-1)*|c|$, so that c affects two simple transformations and not one.
Note 2: if c is affects **neither first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations.
Note 3: if **the same operation** is applied to x in all positions in the equation this operation still should be treated as the first operation, as in $\ln(x - 2)+\sin(x - 2)$.
Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is effected by adding a constant c to y or multiplying it by a constant c, y should be made the subject of the functional equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\frac{1}{c}$, respectively.

Finding a Limit of a Sequence

To find a limit of a sequence (as $k \to \infty$) use the first principles (graphical representation) or

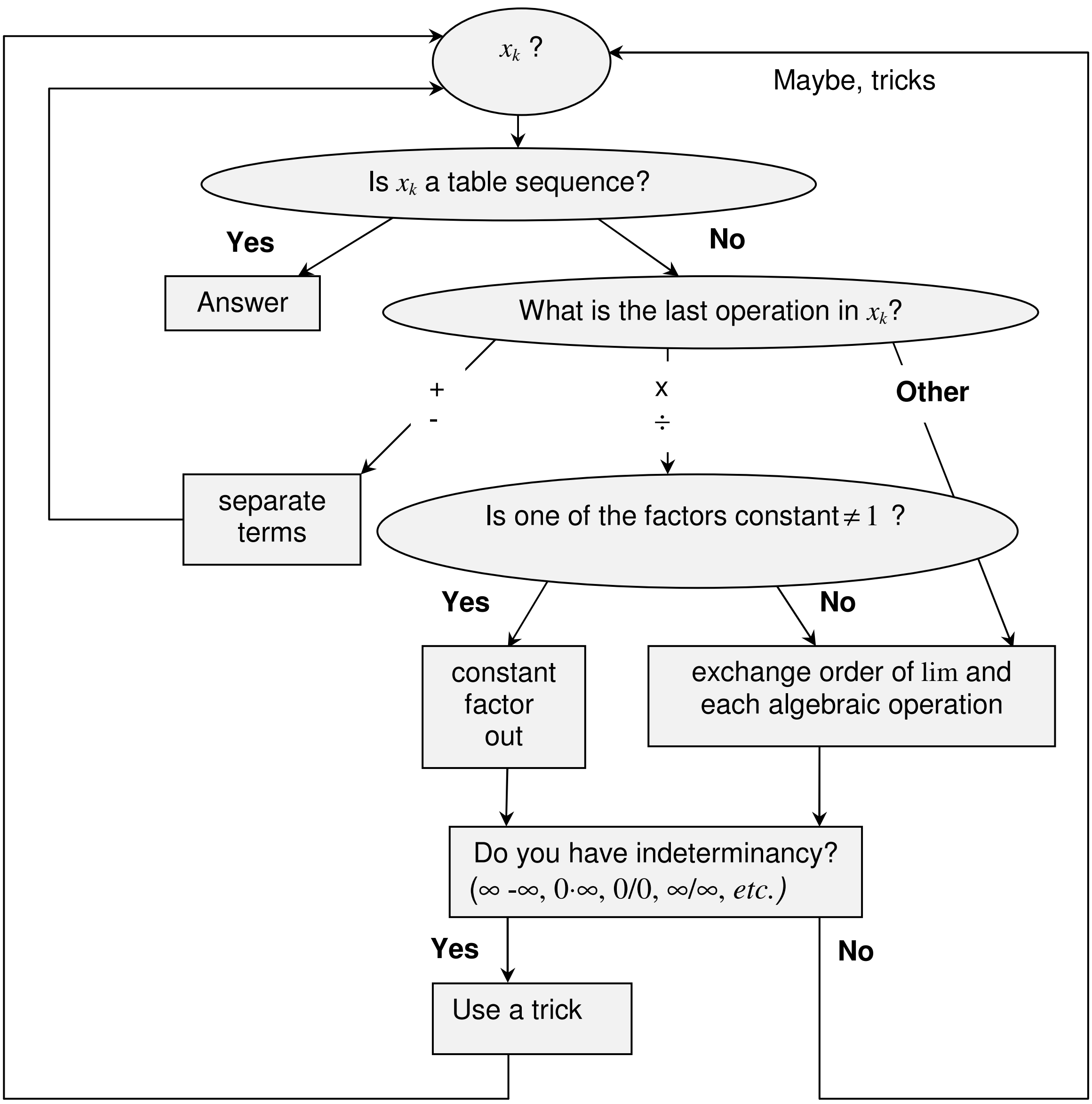
TABLE

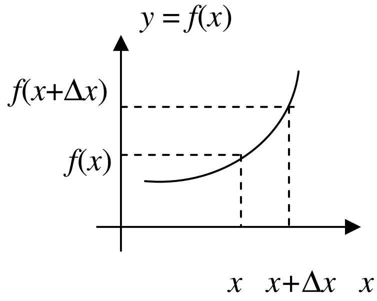
x_k	$\lim x_k$
constant	Constant
k	∞
$\frac{1}{k}$	0
$\frac{1}{a^n}$ $(a>0)$	1
q^n $(\lvert q \rvert < 1)$	0

RULES

1. $\lim \alpha x_k = \alpha \lim x_k$ – *const factor out*
2. $\lim [x_k \pm y_k] = \lim x_k \pm \lim y_k$ – *sum rule (separate terms)*
3. $\lim (x_k\, y_k) = \lim x_k \lim y_k$ – *product rule*
4. $\lim \frac{x_k}{y_k} = \frac{\lim x_k}{\lim y_k}$ – *quotient rule*
5. $\lim f(x_k) = f(\lim x_k)$
   limit and any algebraic operation or simple function can exchange places (commute)

DECISION TREE





Differentiation Summary

To differentiate a function use the first principles

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

or represent roots and fractions as powers, make invisible brackets visible and use

TABLE

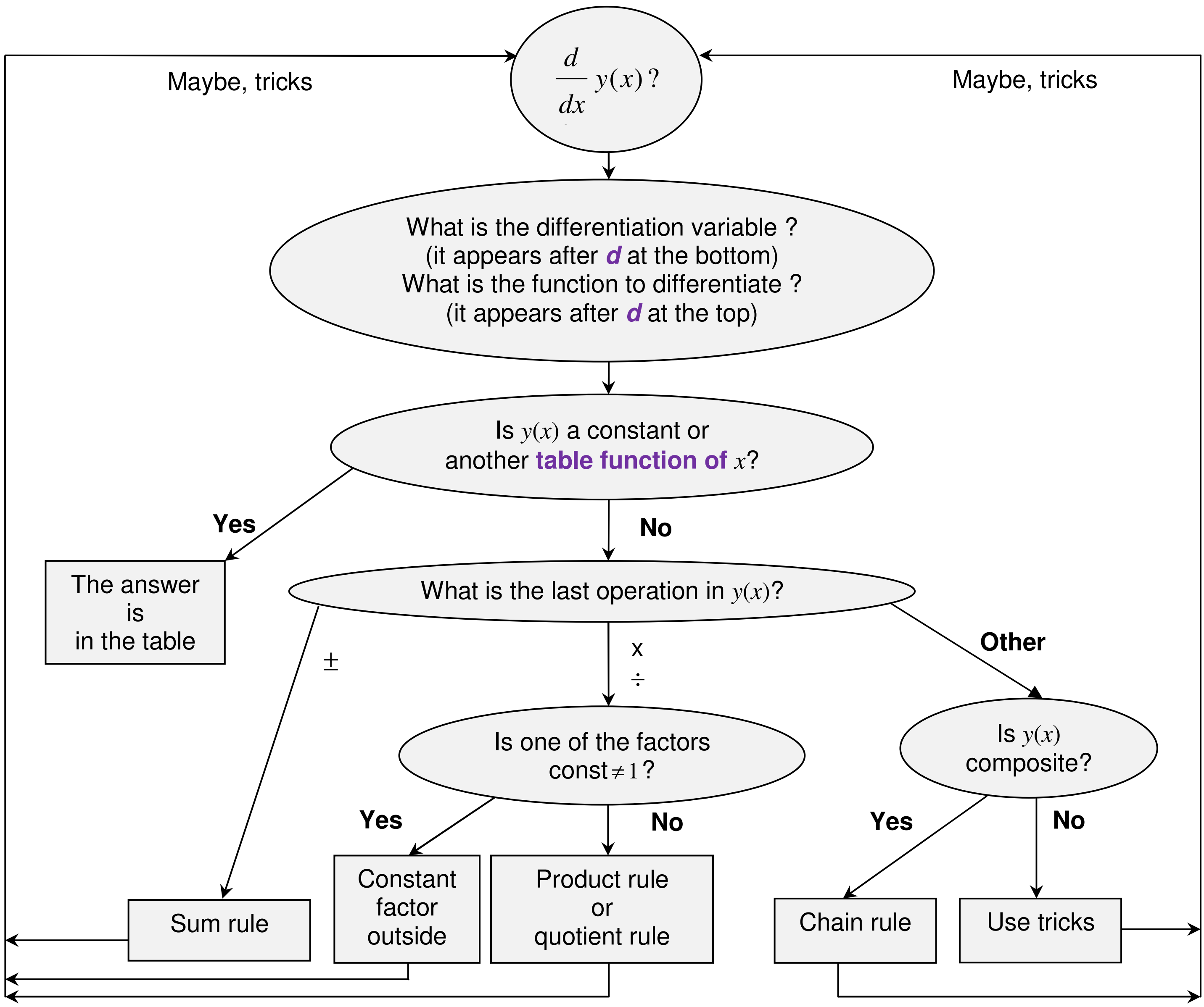
$f(x)$	$\frac{d}{dx} f(x)$
constant	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

RULES

1. $\frac{d}{dx}[\alpha f(x)] = \alpha \frac{df(x)}{dx}$ - *const factor out*
2. $\frac{d}{dx}[f(x) + g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$ – *sum rule (separate terms)*
3. $\frac{d}{dx}[f(x)g(x)] = \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x)$ – *product rule*
4. $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{\frac{df(x)}{dx} g(x) - \frac{dg(x)}{dx} f(x)}{g^2(x)}$ – *quotient rule*
5. $\frac{d}{dx}[f(g(x))] = \frac{df(g(x))}{dg(x)} \frac{dg(x)}{dx}$ – *chain rule*

(decompose, differentiate, multiply)

DECISION TREE



V. GLOSSARY

ABSTRACTION - a general concept formed by extracting common features from specific examples.

ACUTE ANGLE – a positive angle which is smaller than 90^0.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by the highest common DIVISOR.

CIRCLE – a LOCUS of points inside and including a CIRCUMFERENCE.

CIRCUMFERENCE – a LOCUS of points at the same distance from a specific point (called a centre).

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

COMPOSITION – a combination of two or more functions (operations) forming a single new function by applying one to the output of another, e.g. composition of two function '$f\,(\,)$' and '$g\,(\,)$' can be represented as $f\,(g\,(\,))$.

CONCEPT - a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN, e.g. in the EXPRESSION x + a, x is usually understood to represent a VARIABLE and a, a constant (which is independent of this VARIABLE).

If there is only one algebraic CONSTANT, the preferred choice for its algebraic symbol is a. The second choice is b, and the third, c. If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM - a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s,* kg/m^3, £.

DOMAIN – a set of all allowed values of the ARGUMENT.

EQUATION – a mathematical statement involving VARIABLES and the = sign which can be true or false, depending on the values taken by the VARIABLES (which are called UNKNOWNS in this case), e.g. $2x + 3y = 10$ is true when $x = y = 2$ but not when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. $2(a + b)$, $2ab$.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, *a* and *b*.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

FUNCTION – a mathematical OPERATION or a composition of OPERATIONS which establishes a relationship between values of the ARGUMENTS (INDEPENDENT VARIABLES, INPUTS) in its DOMAIN and VALUE OF THE FUNCTION (DEPENDENT variable, output). a function is completely defined only when its domain is defined, e.g. $f(x) = 2x + 3$, x - real' describes a FUNCTION whose SYMBOL is 'f' or '$f(\ \ )$' (no multiplication is implied) and whose ARGUMENT is called 'x'. Its DOMAIN is 'all reals'. Given any value of 'x', you can find the corresponding value of this FUNCTION by first multiplying the given value of 'x' by 2 and then by adding 3 to the result.

FUNCTION SYMBOL – usually a Latin letter, usually from the first part of the alphabet. If there is only one FUNCTION, the preferred choice for its symbol is 'f'. The second choice is 'g', and the third, 'h'. If there are more variables, then one chooses, in order of preference, 'u', 'v' and 'w'. If more FUNCTIONS are involved, we often make use of subscripts and superscripts, upper case and Greek letters.

GENERALISATION - an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH - a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INDEPENDENT VARIABLE – ARGUMENT, INPUT, e.g. given '$y = f(x)$', 'x' is an INDEPENDENT VARIABLE.

INPUT – (value of) ARGUMENT, (value of) INDEPENDENT VARIABLE, e.g. given '$y = f(x)$', 'x' is INPUT; also given '$y = f(2)$', '2' is INPUT.

INTEGER PART – when dividing a positive integer m into a positive integer n, k is the INTEGER PART if it is the largest positive integer producing $k*m \le n$. The REMAINDER is the difference $n - k*m$, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that (if it exists) undoes what the original OPERATION does

LAST OPERATION – see ORDER OF .OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION – an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. '$2x - 3 = 0$' is a LINEAR EQUATION, '2' is a COEFFICIENT in front of the UNKNOWN and '-3' is a FREE TERM.

LOCUS – a set of all points on a plane (or in space) with a specific property.

NECESSARY - ‘A’ is a NECESSARY condition of 'B' IF 'B $\Rightarrow$ A’ (B implies A), so that 'B' cannot take place unless 'A' is satisfied.

NON-DIMENSIONAL QUANTITY - a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. '$2 \ln(x) - 3 = 0$' is a NON-LINEAR EQUATION.

OPERATION (mathematical) – something that can be done to CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which the OPERATIONS must be performed. By convention, the ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for are fractions and functions. Indeed, when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent, and some authors do not bracket ARGUMENTS of elementary FUNCTIONS, such as exp, log, sin, cos, tan *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.* Other OPERATIONS must be performed in the order of decreasing complexity, which is

FUNCTIONS $f(\ )$

POWERS (including inverse operations of roots and logs)

MULTIPLICATION (including inverse operation of division)

ADDITION (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

OUTPUT – (value of) DEPENDENT VARIABLE, (value of the) FUNCITON, e.g. given '$y = f(x)$', 'y' is OUTPUT; also given '$f(x) = 2x + 3$' and '$x = 2$', '7' is OUTPUT (indeed, $2*2+3 = 7$).

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, x, e.g. 'ab' is a PRODUCT, and so is '$(a + b)c$'.

QUOTIENT - a mathematical expression where the last operation is division.

RANGE – the set of all possible values of the function. **HERE**

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the '=' sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SEQUENCE – a function with an INTEGER ARGUMENT.

SIMPLE EQUATION – an EQUATION OF ONE UNKNOWN, LINEAR or NON-LINEAR but such that can be reduced to LINEAR by a simple CHANGE OF VARIABLE.

SIMPLE TRANSFORMATIONS - translation, scaling or reflection - are affected by adding a constant or multiplying by a constant.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT - 'A' is a SUFFICIENT condition of 'B' IF 'A $\Rightarrow$ B' (A implies B), so that if 'A' is satisfied, then 'B' takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is addition, +, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its algebraic symbol is x. The second choice is y and the third, z. If there are more variables, then, one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

VI. STUDY SKILLS FOR MATHS

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours** each week:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.

2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**

3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).

4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

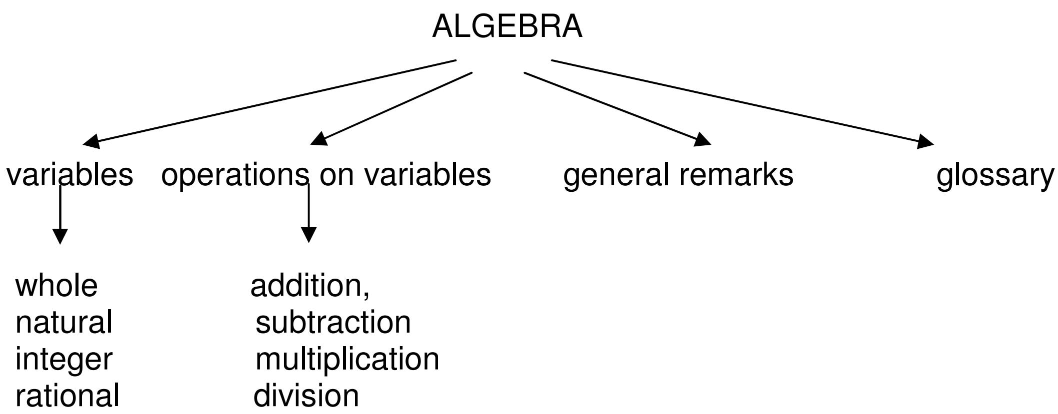
Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

Thus, you can construct the flow chart which looks like



2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.

3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing.** Try this a few times before checking with the notes which step is a problem.

5. Repeat the process for each problem.

6. Repeat the process for each subtopic.

7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.

2. Keep consulting the **Glossary**.

3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.

4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to “make invisible brackets visible”.

2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.

3. The concept of **inverse operation**.

4. The **Decision Tree for Solving Simple Equations**.

5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.

6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real funciton of real variable (real argument). You should know the precise meaning of the word constant (you should always say "constant with respect to (the independent variable) x or t or whatever..."

7. How to do function composition and decomposition using **Order of Operations**

8. How to use graphs

9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos

10. The **Trigonometry Summary**

11. The approximate values of e (≈ 2.71) and π (≈ 3.14)

12. What is j $(=\sqrt{-1})$ and what is j^2 $(=-1)$

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand digram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

15. The **Differentiation Summary**

16. The **Integration Summary**

17. The **Sketching by Simple Transformation Summary**

18. The **Limits Decision Tree** and basic indeterminacies

19. Sketching by analysis

20. The definition of a **mean of a function on an interval**

VII. TEACHING METHODOLOGY (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question - Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft's book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGING AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND SMETHING IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do an exercise yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) - rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education - development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you - in its essence - for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise just a sequence of names, facts or dates there are well established techniques promoted in various books on memory. They suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with this objects. However, this technique will not work with technical information. What you need to establish are much deeper - meaningful - connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: The principles based on the first principles I feel are out of the question at our level in S1 and should be left to S2 as a minimum.

A: These proofs are above the A-level, but are definitely the 1st year level. Everyone should be able to start these proofs on the right note, namely write out the appropriate definitions and then SUBSTITUTE the appropriate functions. Only the students who reached the 1st class level are expected to finish these proofs using various tricks, but everyone should be able to follow the proofs as given by their teachers in Solutions to Exercises and learn how the tricks work. Also, remember, that in real life you will hardly ever differentiate or integrate using differentiation or integration techniques, but remembering the definitions (a derivative is a local slope or a local rate of change and integral is a signed area between a curve and a horizontal axis) might come very handy in your engineering life.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now - in their full glory - LATER, in year 2 and 3, not to-day.

Hope this helps!