Larissa Fradkin

College Algebra – Learn to Love it!

14 September 2013

Table of Contents

I. INTRODUCTION

These notes are based on the lectures delivered by the author to engineering students of London South Bank University over the period of 16 years. This is a University of widening participation, with students coming from many different countries, many of them not native English speakers. Most students have limited mathematical background and limited time both to revise the basics and to study new material. A system has been developed to assure efficient learning even under these challenging restrictions. The emphasis is on systematic presentation and explanation of basic abstract concepts. The technical jargon is reduced to the bare minimum.

Nothing gives a teacher a greater satisfaction than seeing a spark of understanding in the students' eyes and genuine pride and pleasure that follows such understanding. The author's belief that most people are capable of succeeding in - and therefore enjoying - the kind of mathematics that is taught at Universities has been confirmed many times by these subjective signs of success as well as genuine improvement in students' exam pass rates. Interestingly, no correlation had ever been found at the Department where the author worked between the students' qualification on entry and graduation.

The book owns a lot to the authors' students – too numerous to be named here - who talked to her at length about their difficulties and successes, e.g. see Appendix VII on Teaching Methodology. One former student has to be mentioned though – Richard Lunt – who put a lot of effort into making this book much more attractive than it would have been otherwise.

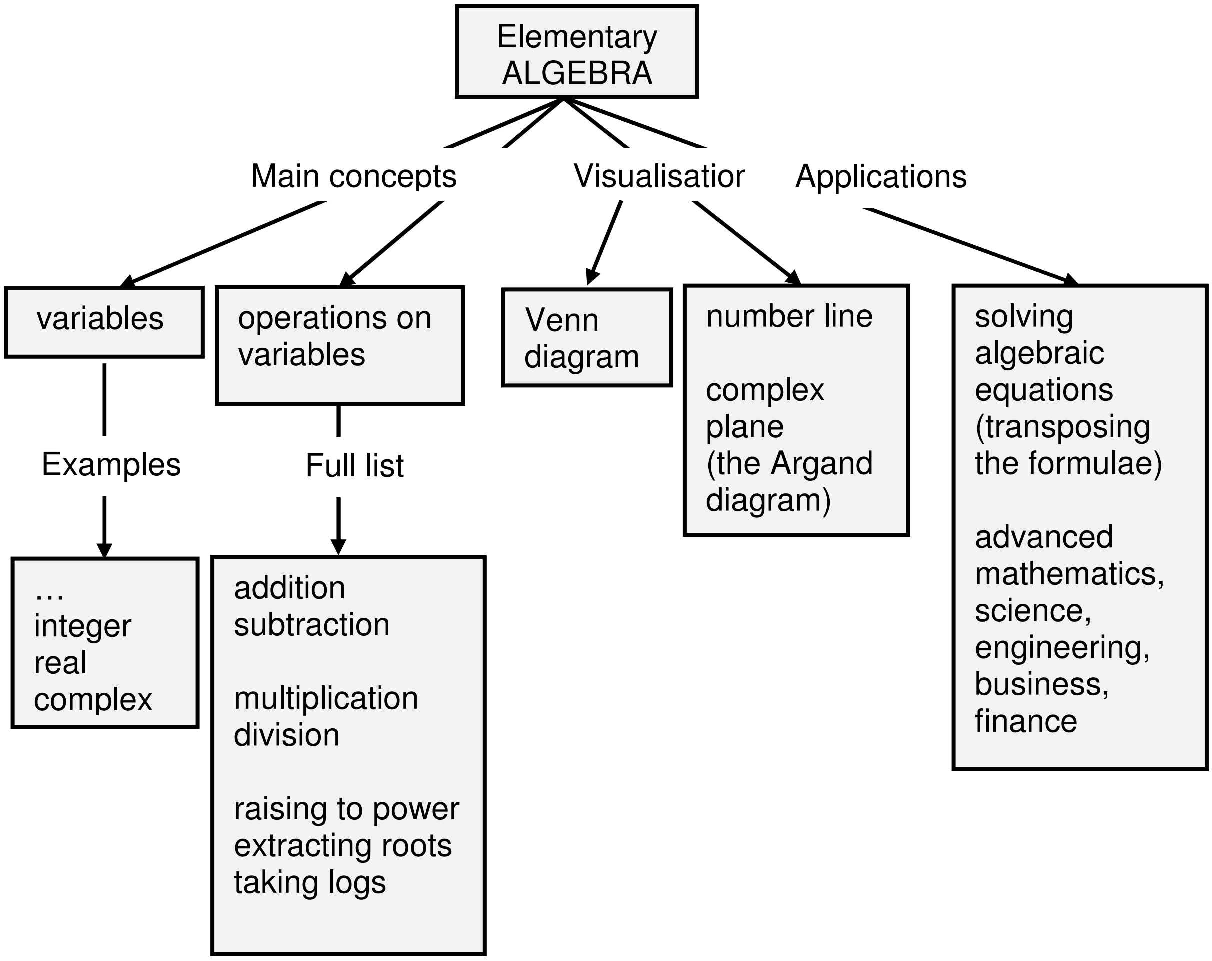
The author can be contacted through her website www.soundmathematics.com. All comments are welcome and teachers can obtain there the copy of notes with answers to questions suggested in the text as well as detailed Solutions to suggested Exercises. The teachers can then discuss those with students at the time of their convenience.

Good luck everyone!

II. CONCEPT MAPS

Throughout when we first introduce a new **concept** (a technical term or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea. Two major topics are covered in this course, Elementary Algebra and Functions.

Here is a **concept map** of Elementary Algebra. It is best to study it before studying any of the Algebra Lectures 1 – 3 and 10 – 12 to understand where it is on the map. The more you see the big picture the faster you learn!



III. LECTURES

We start by introducing a general framework and general concepts used throughout all analytical subjects you are likely to study.

Lecture 1. ALGEBRA: Addition, Subtraction, Multiplication and Division of Rational Numbers

In Elementary Algebra we study **variables** and **operations on variables** which are called algebraic. These concepts are discussed below.

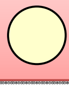
1.1 Variables

A **variable** is an **abstraction** of a quantity.
In algebra, variables are denoted mostly by $a, b, c, d, i, j, k, l, m, n, x, y$ and z.

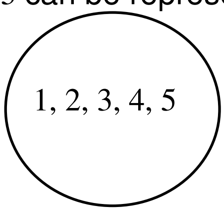
Abstraction is a general concept formed by extracting common features from specific examples. Specific examples of a quantity are time, distance, magnitude of force, current, speed, concentration, profit and so on.

A variable can take any **value** from an allowed **set of numbers**. If a variable represents a dimensional quantity, that is, a quantity measured in dimensional units s, m, N, A, m/s, kg/m^3, £… each value has to be multiplied by the corresponding unit. Otherwise the variable is called non-dimensional.

Diagrammatically any set can be represented (visualised) as a circle (this circle is called a **Venn diagram**). It might help you to think of this circle as a bag containing all elements of the set.



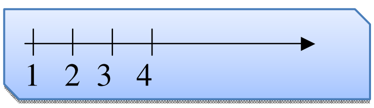
Example: A set of numbers 1, 2, 3, 4, 5 can be represented using the Venn diagram



1.2 Variables and operations on variables

1.2.1 Variables: Whole numbers

Whole numbers are 1, 2, 3, ..., that is, the numbers used to count (three dots stand for ***etc.***, that is, **"and so on"**). **The set of whole numbers can be visualised graphically using the number line:**



A **number line** is a straight arrowed line. A point representing number 1 is chosen arbitrarily on this line, and so is the unit distance between the points representing numbers 1 and 2. All further neighbouring points are separated by the same unit distance and represent numbers 3, 4 *etc.* The arrow reminds us that the further the point is positioned to the right the greater the number.

Whole numbers are usually denoted by letters i, j, k, l, m, n.

1.2.2 Operations: Addition

Operations on variables are things you can do with variables.

Addition is the first and simplest algebraic operation. Its symbol is the + sign (read as "plus"). **Addition** of a whole number n to a whole number m **can be visualised using the number line**:

- find a point on the number line that represents number m,
- move along the number line n units to the right.

Addition is called a **direct operation** to emphasise two facts:

1. we just define (declare) what the result of addition is and
2. addition of whole numbers results in a whole number.

We now introduce the laws of addition which are easy to verify (but not prove) by substituting whole numbers. We verbalise the laws in a way that helps us to apply them when required to perform algebraic manipulations.

Law 1: $a + b = b + a$

Terminology:

a and b are called **terms** (expressions that are being added),
$a + b$ is called a **sum** (expression in which the last operation is addition).

Law 1 verbalised: order of terms does not matter.

Law 2: $(a + b) + c = a + (b + c)$.
Law 2 verbalised: knowing how to add up two terms we can add up three terms, four terms *etc.* (add up any two, add the result to any of the remaining terms, repeat the operation until all terms are used up).

1.2.3 Operations: Subtraction

The symbol of subtraction is the – sign (read as "minus"). Subtraction is an **inverse** (opposite) **operation** to addition. This means that it is defined via addition:

Definition: $a - b = x$: $x + b = a$.

in this context **:** means **"such that"**

The definition implies that we have the following relation between addition and subtraction:

$a + b - b = a$ (subtraction undoes addition)

$a - b + b = a$ (addition undoes subtraction)

1.2.4 Variables: Integers

Subtraction is the first inverse operation we encounter and as with many other inverse operations considered later its application might cause a difficulty: subtracting a whole number from a whole number does not always result in a whole number. However, using specific examples of debt and temperature it makes sense to say that subtraction introduces new types of numbers, **0** and **negative whole numbers**:

1. $a - a = 0$
2. $a - b = -(b - a)$ if $b > a$

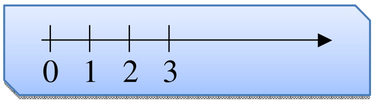
this symbol means **"greater than"**

Note: the – sign between variables or numbers is a symbol of subtraction while the – sign in front of a number tells us that the number is negative. This might be a bit confusing but once the convention is grasped it becomes very convenient!

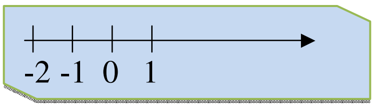
Question: is $-a$ positive, negative or zero?
Answer:

We can now introduce wider number sets than the set of whole numbers:

Natural numbers are $0, 1, 2, \ldots$ The set of natural numbers can be visualised graphically as points on the number line:



Integers are ..., $-2, -1, 0, 1, 2,$ ... The set of integers can be visualised graphically in a similar manner:



An **integer variable** is a variable that takes integer values. Integer variables are usually denoted by letters $i, j, k, l, m, n.$

1.2.5 Operations: Addition and Subtraction (ctd.)

We can now introduce further laws of addition:

Law 3: $a + 0 = a.$
Law 3 verbalised: The zero term can always be dropped or put in.

Law 4: for each a there exists one **additive inverse** $-a$: $a + (-a) = 0.$
Law 4 verbalised: every number has an additive inverse.

Laws of addition can be used to justify the rules given below:

Rules

1. $+(b + c) = b + c$ (since $+(b + c) = 0 + (b + c) = (0 + b) + c = b + c$)
Rule 1 verbalised: when removing brackets with + in front just copy the terms inside the brackets.

2. $+a + b = a + b$ (since $+a + b = 0 + a + b = (0 + a) + b$)
Rule 2 verbalised: the + sign in front of the first term can be dropped or put in.

3. $-(a) = -a$ (since additive inverse of a is $-a$)

4. $-(-a) = +a$ (since additive inverse of $-a$ is a)
Rules 3 and 4 verbalised: when removing brackets with the minus sign in front copy each term inside the brackets but with the opposite sign.

Subtraction of an integer n from a number m can be visualised using the number line:

- choose the point on the number line that represents number m
- if n is a whole number move along the number line to the left by n units
- if n is zero there should be no movement,
- if n is a negative integer move along the number line to the right by $-n$ units (it could not be moving to the left, otherwise there would be no difference between subtracting positive integers and subtracting negative integers!)

Note: Using Rule 1 we can write

this sign means **"implies"**

$$a + (-b) = a - b \quad \Rightarrow \quad a - b = a + (-b).$$

that is, a **difference** (expression in which the last operation is subtraction) can be turned into a sum (expression in which the last operation is addition). This becomes useful in some algebraic manipulations.

Question: How many terms are there in expression $3 - 2a$ and what are they?
Answer:

1.2.6 Operations: Multiplication

Multiplication is the second direct algebraic operation. Its symbols are the $\times$ sign or else the $\cdot$ sign. When symbols of variables are put next to each other or else when they are put next to a number, the absence of any operational sign also indicates multiplication. In other words we can write:

$$ab = a \cdot b = a \times b, \;\; 2b = 2 \cdot b = 2 \times b$$

For whole numbers n, the result of multiplication of a by n is

$$a \cdot n = \underbrace{a + \dots\dots\dots\dots + a}_{n \text{ equal terms}}$$

Thus, multiplication by a whole number n is shorthand for addition of n equal terms.

Example: $5a$ is a much shorter expression than $a + a + a + a + a$.

Multiplication is a **direct operation** in the sense that we just define (declare) what a product of a number and a whole number is and also, in the sense that a product of integers is an integer. We now introduce the laws of multiplication which are easy to verify (but not prove) by substituting whole numbers. As before, we verbalise them so as to make algebraic manipulations an easier task.

Law 1: $a \cdot b = b \cdot a$.

> **Terminology**:
>
> a and b are called **factors** (expressions that are multiplied),
> *ab* is called a **product** (expression in which the last operation is multiplication).

Law 1 verbalised: order of factors does not matter.

Law 2: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

> **Convention:**
>
> $abc = (ab)c, \quad a(-bc) = -abc$.

Law 2 verbalised: knowing how to multiply two factors can multiply three, four *etc.*

Law 3: $a(b + c) = ab + ac$

$\longrightarrow$ Removing brackets
$\longleftarrow$ Factoring

Law 3 verbalised:
Left to right: we can turn a product into a sum, that is, can change the order of operation from AM (Addition, Multiplication) to MA (Multiplication, Addition).
Right to left: we can turn a sum into a product, that is, can change the order of operation from MA (Multiplication, Addition) to AM (Addition, Multiplication).

Law 4: $a \cdot 0 = 0$.
Law 4 verbalised:
Left to right: any number times 0 is 0.
Right to left: 0 can be represented as a product of 0 and any other number.

Law 5: $a \cdot 1 = a$.
Law 5 verbalised: factor 1 can be dropped or put in.

The multiplication and addition laws can be used to deduce the following useful rules:

Rules

1. $(a+b)(c+d) = ac + ad + bc + bd$ - **SMILE RULE**

The rule can be proved using Multiplication Law 3.

Question: Is the order of terms and factors important?
Answer:

Memorising the **SMILE RULE** is advisable, because it makes checking results easier. It can be extended to multiplying any number of sums with any number of terms.

2. $(-1)\times n = -n$.

Justification:

The answer should be n with a sign and it cannot be $+n$, because then there would be no difference between $(-1)\times n$ and $1\times n$. So the answer should be $-n$. (Note that if n is a whole number, we can produce another proof: if n is a whole number $(-1)\times n$ $= -1+(-1)+\ldots+(-1)$ - sum of n (-1)s - and so $(-1)\times n = -n$.)

Rule 2 verbalised:
Left to right: multiplying by -1 is equivalent to changing sign.
Right to left: the minus sign in front of a term is a shorthand for $(-1)\times$.

1.2.7 Operations: Division

If writing is restricted to just one line of text, the symbol of division is the ÷ sign or else / sign. More often than not one uses the – sign, with one expression on top and another, at the bottom. It should not be confused with the minus sign. Division is an **inverse** (opposite) **operation** to multiplication. This means that it is defined via multiplication:

Definition: $\frac{a}{b} = x: \; xb = a$

The definition implies that we have the following relations between multiplication and division:

$\frac{ab}{b} = a$ (division undoes multiplication)

$\frac{a}{b}b = a$ (multiplication undoes division)

Terminology:

a – **numerator** (expression divided),
b – **denominator** (expression by which numerator is divided),
$\frac{a}{b}$ – **quotient, ratio** (expression in which the last operation is division).
A quotient is called a **proper fraction** if a, b are whole numbers and $a < b$. It is called an **improper fraction** if a, b are whole numbers and $a \geq b$.

this sign means **"does not equal to"**

Convention:

$2\frac{1}{2} \neq 2 \cdot \frac{1}{2}$, $2\frac{1}{2} = 2 + \frac{1}{2}$, $2\frac{3}{2} \neq 2 + \frac{3}{2}$ but rather $2\frac{3}{2} = 2 \cdot \frac{1}{2}$

So, when adding a whole number and a proper fraction just put this fraction next to the whole number. When multiplying a whole number by an improper fraction or adding them up put the appropriate operation sign between them to avoid confusion.

1.2.8 Variables: Rational Numbers

Division is the second inverse operation we encounter and again its application might cause a difficulty: dividing an integer by an integer does not always produce an integer. To give a simple example, dividing 2 pies between 2 people, each gets two pies ($2/2 = 1$, in this case division of two whole numbers produces a whole number). However, dividing 1 pie between two people can be only achieved by cutting this pie into two portions ($1/2$ is not a whole number, but a new type of number called **rational**).

Definition: A **rational number** is a number $\frac{m}{n}$, where m and $n \neq 0$ are integers.

Division by zero is undefined!

Proof

Indeed, **assume** that it is defined, that is, there exists a number x such that $\frac{a}{0} = x$.

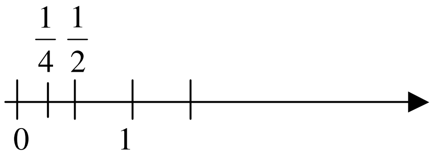
Let $a \neq 0$. Then $0 \times x = a$. The number on the left of the = sign is 0 and the number on the right is not. Hence we arrived at a contradiction.
Let now $a = 0$. Then $0 \times x = 0$. However, this is true for all numbers and not just one.

Hence the **assumption** (that there exists a number x, such that $\frac{a}{0} = x$) **is invalid**.

Question: Is 2 a rational number? Explain your answer.

Answer:

Rationals can be visualised graphically using the number line:



Question: How many rationals are there between 0 and 1?
Answer:

A **rational variable** is a variable that takes rational values.

1.2.9 Operations: Multiplication and Division (ctd.)

Law 6: For each $a \neq 0$ there exists one **multiplicative inverse** $\frac{1}{a}$: $a \cdot \frac{1}{a} = 1$.
Law 6 verbalised: every number but 0 has a multiplicative inverse.

The addition and multiplication laws can be used to deduce useful rules described below:

Rules

1. $\frac{a}{b} \cdot n = \frac{an}{b}$

Rule 1 verbalised: any product (an expression where the last operation is multiplication) can be turned into quotient (an expression in which the last operation is division) and ***vice versa*** (that is, **"the other way around"**).

Note: Using Rule 1 right to left, $\frac{a}{b} = a \times \frac{1}{b}$, that is, a quotient (expression where the last operation is division) can be turned into a product (expression in which the last operation is multiplication).

Question: How many factors are there in expression $\frac{a}{3}$ and what are they?

Answer:

2. $\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$

Convention:

1. Division prescribed with the slash line / is performed before division prescribed with the horizontal line.

2. If only horizontal lines are used to prescribe division the shorter lines take precedence.

3. $\frac{an}{bn} = \frac{a \cdot \cancel{n}}{b \cdot \cancel{n}} = \frac{a}{b}$ - **CANCELLATION** (in this context, the slash means **cross out** and **not divide**)

Rule 3 verbalised:
Left to right: if first multiplying by a number/expression and then dividing by it this number/expression can be cancelled.
Right to left: can multiply both numerator and denominator by the same non-zero number/expression without changing the quotient.

4. $\frac{a/b}{c/d} = \frac{ad}{cb}$ - **FLIP RULE**

Rules 2 and 4 verbalised:
Left to right: a multi-storey fraction can always be turned into a two-storey fraction.
Right to left: a two-storey fraction can always be turned into a multi-storey fraction.

5. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

Note: $\frac{a+b}{c+d} = (a + b)/(c + d)$ (if a quotient is presented on one line and there is more than one term in the numerator or denominator they have to be bracketed).

6. $\frac{a}{b}\backslash{\times d} + \frac{c}{d}\backslash{\times b} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd}$ - **COMMON DENOMINATOR RULE**

This line means **"apply the same operation to top and bottom"**.
What follows this line is the operation to be applied.

Law 5 and 6 verbalised: we can turn a sum of quotients into a quotient and *vice versa.* Another way of putting it: we can change the order of operation from DA (Division, Addition) to AD (Addition, Division) and *vice versa*.

1.3 General remarks

1. All the laws and rules introduced above can be checked (but not proved!) by **substituting** whole numbers for letters (that is, by putting whole numbers in place of letters). If you are not sure whether you remember a law or rule correctly, this type of substitution might jog your memory. **Checking by substitution is not 100 % safe, since the "law" you invent might by chance be satisfied by a couple of whole numbers but not all of them.** This means that a check of that nature will minimise the chance of a mistake but not eliminate it.

2. It is easier to remember the above laws and rules from **left to right**. However, you have to be able to use them from **right to left** with similar ease.

3. Note that while the phrases "integer variable" or "rational variable" are quite conventional nobody talks of a "whole variable" or "natural variable". The mathematical language can be just as inconsistent as the natural language!

4. All laws and rules introduced above are first introduced for whole numbers. They can be **generalised** (that is, declared to apply) to wider sets of numbers too. Checking the consistency of these generalisations lies outside the scope of these notes. We just state that above laws and rules of addition and multiplication apply to all rational numbers.

5. If an expression involves more than one operation performing these operations in different order can produce different results. It is important to memorise the following convention:

Order of Operations (OOO)

1. Make implicit (invisible) brackets visible
(everything raised and everything lowered with respect to the main line is considered to be bracketed)
2. Perform operations in brackets {[()]} first (inside out)

OOO →

Operations in Brackets including implicit (invisible)	Multiplication	Addition
	$\times$	+
	$\div$	-

.
Note: talking about order of operations we treat both + and – signs as Addition symbols and both $\times$ and $\div$ signs as Multiplication symbols. Similar rules introduced in school make unnecessary distinctions which suggest that division should be performed before multiplication and addition before subtraction. Not so!

1.4 Glossary of terms introduced in this Lecture

An **abstraction** is a general concept formed by extracting common features from specific examples.

A **diagrammatic representation** is a very general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

A **generalisation** is an act of introducing a general concept or rule by extracting common features from specific examples.

A **graphical representation** is a specific visualisation tool, a pictorial representation of a particular set or relationship.

A **sum** is a mathematical expression where the last operation is addition.

A **difference** is a mathematical expression where the last operation is subtraction.

Terms are expressions you add. They are separated by + sign(s).

A **product** is a mathematical expression where the last operation is multiplication.
A **quotient** is a mathematical expression where the last operation is division.

Factors are expressions you multiply. They are separated by multiplication sign(s).

Note 1: Our usage of words *term* and *factor* is not universal. Mathematicians also use words *addend* and *summand* for the *term*. Some talk of *multipliers* and *multiplicands* rather than *factors*. Engineers use words *term* and *factor* interchangeably - very confusing! In these notes we use them only in the sense described above. This allows us to produce very short explanations.

Note 2: From now own you are expected to create your own glossary for each lecture. Remember – the words and phrases in bold red introduce new concepts and conceptual ideas. Do not forget to keep using the Glossary Appendix.

1.5 Historical notes

“We learn to count at such an early age that we tend to take the notion of abstract numbers for granted. We know the word "two" and the symbol "2" express a quantity that we can attach to apples, oranges, or any other object. We readily forget the mental leap required to go from counting specific things to the abstract concept of number as an expression of quantity.

Abstract numbers are the product of a long cultural evolution. They also apparently played a crucial role in the development of writing in the Middle East. Indeed, numbers came before letters.”

http://www.maa.org/mathland/mathland_2_24.html

Negative numbers appear for the first time in history in the *Nine Chapters on the Mathematical Art* (Jiu zhang suan-shu), which in its present form dates from the period of the Han Dynasty (202 B.C. – A.D. 220), but may well contain much older material. The Nine Chapters used red counting rods to denote positive coefficients and black rods for negative. (This system is the exact opposite of contemporary printing of positive and negative numbers in the fields of banking, accounting, and commerce, wherein red numbers denote negative values and black numbers signify positive values). The Chinese were also able to solve simultaneous equations involving negative numbers.

For a long time, negative solutions to problems were considered "false". In Hellenistic Egypt, Diophantus in the third century A.D. referred to an equation that was equivalent to $4x + 20 = 0$ (which has a negative solution) in *Arithmetica*, saying that the equation was absurd.

The use of negative numbers was known in early India, and their role in situations like mathematical problems of debt was understood. Consistent and correct rules for working with these numbers were formulated. The diffusion of this concept led the Arab intermediaries to pass it to Europe.

The ancient Indian *Bakhshali Manuscript*, which Pearce Ian claimed was written some time between 200 B.C. and A.D. 300, while George Gheverghese Joseph dates it to about A.D. 400 and no later than the early 7th century, carried out calculations with negative numbers, using "+" as a negative sign.

During the 7th century A.D., negative numbers were used in India to represent debts. The Indian mathematician Brahmagupta, in *Brahma-Sphuta-Siddhanta* (written in A.D. 628), discussed the use of negative numbers to produce the general form quadratic formula that remains in use today. He also found negative solutions of quadratic equations and gave rules regarding operations involving negative numbers and zero, such as *"A debt cut off from nothingness becomes a credit; a credit cut off from nothingness becomes a debt. "* He called positive numbers "fortunes," zero "a cipher," and negative numbers "debts." During the 8th century A.D., the Islamic world learned about negative numbers from Arabic translations of Brahmagupta's works, and by A.D. 1000 Arab mathematicians were using negative numbers for debts.

In the 12th century A.D. in India, Bhaskara also gave negative roots for quadratic equations but rejected them because they were inappropriate in the context of the problem. He stated that a negative value is "*in this case not to be taken, for it is inadequate; people do not approve of negative roots.*"

Knowledge of negative numbers eventually reached Europe through Latin translations of Arabic and Indian works. European mathematicians, for the most part, resisted the concept of negative numbers until the 17th century, although Fibonacci allowed negative solutions in financial problems where they could be interpreted as debits (chapter 13 of *Liber Abaci*, A.D. 1202) and later as losses (in *Flos*).

In the 15th century, Nicolas Chuquet, a Frenchman, used negative numbers as exponents and referred to them as “absurd numbers.”

In A.D. 1759, Francis Maseres, an English mathematician, wrote that negative numbers "darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple". He came to the conclusion that negative numbers were nonsensical.

In the 18th century it was common practice to ignore any negative results derived from equations, on the assumption that they were meaningless.
http://en.wikipedia.org/wiki/Negative_and_non-negative_numbers#History

It has been suggested that the concept of irrationality was implicitly accepted by Indian mathematicians since the 7th century BC, when Manava (c. 750–690 BC) believed that the square roots of numbers such as 2 and 61 could not be exactly determined, but such claims are not well substantiated and unlikely to be true.

The first proof of the existence of irrational numbers is usually attributed to a Pythagorean (possibly Hippasus of Metapontum), who probably discovered them while identifying sides of the pentagram. The then-current Pythagorean method would have claimed that there must be some sufficiently small, indivisible unit that could fit evenly into one of these lengths as well as the other. However, Hippasus, in the 5th century BC, was able to deduce that there was in fact no common unit of measure. His reasoning is as follows:

- The ratio of the hypotenuse to an arm of an isosceles right triangle is $a{:}b$ expressed in the smallest units possible
- By the Pythagorean theorem: $a^2 = 2b^2$.
- Since a^2 is even, a must be even as the square of an odd number is odd.
- Since $a{:}b$ is in its lowest terms, b must be odd.
- Since a is even, let $a = 2y$.
- Then $a^2 = 4y^2 = 2b^2$
- $b^2 = 2y^2$ so b^2 must be even, therefore b is even.
- However we asserted b must be odd. *Here is the contradiction.*

http://en.wikipedia.org/wiki/Irrational_number

1.6 Instructions for self-study

- **Study Lecture 1 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Simplify

a) 0×1

b) 0×10

c) $0\times x$

d) $0\times(x-1)$

e) 1×1

f) 1×10

g) $1\times x$

h) $1\times(x-1)$

Q2. Solve

a) $0=\frac{x}{2}$

b) $\frac{x-2}{2}=0$

c) $\frac{1}{x}=1$

d) $\frac{1}{x-2}=1$

Q3. Remove brackets

a) $-(2-a)$

b) $+(2+a-b)$

c) $-(2+a-b)$

d) $-(2-(a-b))$

e) $(a+b)(a+b)$

f) $(a+b)(a-b)$

Q4. Factorise

a) $2-6a$

b) $\frac{1}{3}-\frac{1}{9}b$

c) $\frac{1}{3} - \frac{1}{9}b + 2c$

d) $(a + b)c + (a + b)d$

Q5. Turn into a one storey fraction:

a) $\frac{2}{6/3}$

b) $\frac{ab}{a/b}$

Q6. Add fractions:

a) $\frac{1}{2} + \frac{1}{3}$

b) $\frac{1}{x-1} + \frac{1}{x-2}$

c) $\frac{1}{x-1} + \frac{1}{x+1}$

If you need more exercises (particularly on addition of fractions) you can use the following sites: **http://helm.lboro.ac.uk/documents/1_4_arthmtic_algebraic_fractns.pdf**
http://www.mathtutor.ac.uk/viewdisks.php

Lecture 2. Applications of Elementary ALGEBRA: Solving Simple Equations

2.1 Revision: Factorisation

The Multiplication Law 3 allows us to change the order of operations of multiplication and addition. When going from left to right (that is, removing or expanding the brackets), a product is turned into a sum. When going from right to left (that is, factoring), the sum is turned into a product

$$ab + ac = a(b + c) \tag{2.1}$$

Question: What operations are implied in the LHS (left-hand side) of (2.1) and in what order?
Answer:
Question: What would you call the expression in LHS of (2.1) and why?
Answer:
Question: What are the terms in in the LHS of (2.1)?
Answer:

Question: What operations are implied in the RHS (right-hand side) of (2.1) and in what order?
Answer:
Question: What would you call the expression in RHS of (2.1) and why?
Answer:
Question: What are the terms in the RHS of (2.1)?
Answer:

Note that you can turn a sum into a product using any factor a. It does not have to appear as a common factor in all terms in the sum. This is a general recipe for finding b and c:

ab/a   ac/a

$$ab + ac = a(b + c)$$

To verbalise, we can get the first term b in the RHS of (2.1) by dividing by a the first term ab in the LHS and we can get the second term c in the RHS by dividing by a the second term ac in the LHS.

Examples: Factorise and check your answer by removing brackets

1. $3u + 9v = 3(u + 3v)$
2. $xu + 3xv = x(u + 3v)$
3. $(x - 2)u + 3(x - 2)v = (x - 2)(u + 3v)$
4. $(-x + 1)u + 3(-x + 1)v = (-x + 1)(u + 3v)$
5. Find x and y in $s - 1 + \dfrac{3}{s+1} = (s-1)(x + y)$

Solution

$$x = 1,\ \ y = \frac{3}{s^2 - 1}$$

2.2 Revision: Adding fractions

Adding fractions we change the order of operations of addition and division, that is, turn a sum of fractions into one fraction.

Examples: Turn into one fraction

This sign means "**apply the same operaton to both top and bottom**". The opeartion is indicated after the sign.

1. $\frac{1}{2}+\frac{1}{3}$. **Solution:** $\frac{1}{2}\backslash 3+\frac{1}{3}\backslash 2=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$

2. $\frac{1}{x-2}+\frac{1}{3}$. **Solution:** $\frac{1}{x-2}\backslash 3+\frac{1}{3}\backslash (x\text{-}2)=\frac{3}{3(x-2)}+\frac{x-2}{3(x-2)}=\frac{3+x-2}{3(x-2)}=\frac{x+1}{3(x-2)}$

3. $\frac{4}{5}-\frac{2}{3}$. **Solution:** $\frac{4}{5}\backslash 3-\frac{2}{3}\backslash 5=\frac{12}{15}-\frac{10}{15}=\frac{2}{15}$

4. $\frac{2u}{u-2}-\frac{2}{3}$.

Solution: $\frac{2u}{u-2}\backslash 3-\frac{2}{3}\backslash (u\text{-}2)=\frac{6u}{3(u-2)}-\frac{2(u-2)}{3(u-2)}=\frac{6u-2u+4}{3(u-2)}=\frac{4u+4}{3(u-2)}=\frac{4}{3}\frac{u+1}{(u-2)}=1\frac{1}{3}\frac{u+1}{(u-2)}$

2.3 Decision Tree for Solving Simple Equations

Equations are mathematical statements that involve at least one **unknown** variable and the equality sign, =. This means that equations state that the mathematical **expression** in the **LHS** (left-hand side of the equation, to the left of the = sign) is the same as the mathematical **expression** in the **RHS** (right-hand side of the equation, to the right of the = sign). For some values of the unknown these statements are false and for others they are true. The equations are different from **mathematical formulae** which also contain variables and the equality sign, =, but are always true.

Algebraic equations involve only algebraic operations on the unknown(s). By convention the first choice for the symbol of an algebraic unknown is x. If a **value of** x (a number substituted for x) turns an equation into a true statement. This value is called **solution of the equation**. **To solve an algebraic equation** means to find all values of x that turn this equation into a true statement.

Simple equations are equations that can be simply rearranged so that only one term contains the unknown.

Question: What simple manipulations allow you to rearrange algebraic equations so that only one term contains the unknown?

Answer:

Note: After solving an equation with the unknown in a denominator make sure that the value you found does not turn this denominator into zero. If it does it might not be a solution of the original equation!

Once you rearranged the equation so that only one term contains the unknown - conventionally in the LHS - you can use **The Decision Tree for Solving Equations with Only One Term Containing the Unknown** which is presented in figure 2.1. This Decision Tree is based on the idea that to solve such equation we have to keep **applying operations** which are **inverse to the last operation** on the unknown **to both sides** of the equation - **until the unknown becomes the subject of the equation** (stands on its own, no operation are applied to it anymore). The idea is very general and is used to solve all equations that appear in maths and science and contain the unknown only in one term. If the unknown is present in more than one term but it is possible to use mathematical manipulations to reduce the equation to the form with the unknown present only in one term the equation is called simple. In this case we should use a more general Decision Tree presented in figure 2.2.

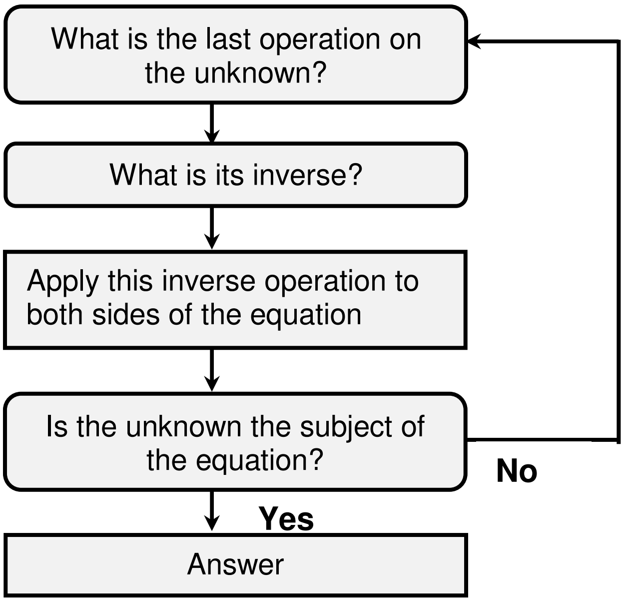


Figure 2.1. Decision Tree for Solving Equations with Only One Term Containing the Unknown

Examples:

Solve the following equations using the Decision Tree for Solving Simple Equations:

1. $x - 1 = 5$

Solution

Using the Decision Tree for Solving Equations with Only One Term Containing the Unknown, the last operation on the unknown x is subtraction, specifically, -1. The inverse operation to -1 is $+1$. Apply this operation to both sides of the equation:

$x - 1 = 5 \;/ +1$

This sign means "**apply the same operation to both sides of the equation**." The opeartion is indicated after the sign.

$x = 6$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct. Consider the LHS, substitute 6 for x, $6 - 1 = 5$. Since LHS = RHS the solution is correct.

2. $2x - 3 = x - 5$

Solution

Using the Decision Tree for Solving Simple Equations, we first collect the like terms: subtracting x from both sides of the equation we get

$$x - 3 = -5$$

The last operation on the unknown x is subtraction, specifically, -3. The inverse operation to -3 is $+3$. Apply this operation to both sides of the equation:

$$x - 3 = -5 \;/ +3$$
$$x = -2$$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct. Consider the LHS, substitute -2 for x, $2 \cdot (-2) - 3 = -4 - 3 = -7$. Consider the RHS, substitute -2 for x, $-2 - 5 = -7$. Since LHS = RHS the solution is correct.

3. $\frac{1}{v-2} + \frac{1}{3} = 0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of The equation by the denominator containing the unknown v:

$$1 + \frac{1}{3}(v - 2) = 0$$

Removing brackets and collecting the like terms,

$$\frac{1}{3}v + \frac{1}{3} = 0$$

The last operation on the unknown v is addition, specifically, $+\frac{1}{3}$. The inverse operation to $+\frac{1}{3}$ is $-\frac{1}{3}$. Apply this operation to both sides of the equation:

$$\frac{1}{3}v + \frac{1}{3} = 0 \;/ -\frac{1}{3}$$

$\frac{1}{3}v = -\frac{1}{3}$

The last operation on the unknown v is multiplication, specifically, $\cdot\frac{1}{3}$. The inverse operation to $\cdot\frac{1}{3}$ is $\cdot 3$. Apply this operation to both sides of the equation:

$v = -1$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct. Consider the LHS, substitute -1 for v, $\frac{1}{-1-2}+\frac{1}{3}=0$. Since LHS = RHS the solution is correct.

4. $\frac{2u}{u-2}+\frac{2}{3}=0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of the equation by the denominator containing the unknown u:

$2u+\frac{2}{3}(u-2)=0$

Removing brackets and collecting the like terms,

$\frac{8}{3}u-\frac{4}{3}=0$

The last operation on the unknown u is subtraction, specifically, $-\frac{4}{3}$. The inverse operation to $-\frac{4}{3}$ is $+\frac{4}{3}$. Apply this operation to both sides of the equation:

$\frac{8}{3}u-\frac{4}{3}=0 \; / \; +\frac{4}{3}$

$\frac{8}{3}u=\frac{4}{3}$

The last operation on the unknown u is multiplication, specifically, $\cdot\frac{8}{3}u$. The inverse operation to $\cdot\frac{8}{3}$ is $\cdot\frac{3}{8}$. Apply this operation to both sides of the equation:

$\frac{8}{3}u=\frac{4}{3} \; / \; \cdot\frac{3}{8}$

$u=\frac{1}{2}$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct. Consider the LHS,

substitute $\frac{1}{2}$ for u, $\dfrac{2\cdot\frac{1}{2}}{\frac{1}{2}-2}+\frac{2}{3}=\dfrac{1}{\frac{1}{2}-\frac{2}{1}}+\frac{2}{3}=\dfrac{1}{\frac{1}{2}-\frac{4}{2}}+\frac{2}{3}=\dfrac{1}{-\frac{3}{2}}+\frac{2}{3}=-\frac{2}{3}+\frac{2}{3}=0$.

Since LHS = RHS the solution is correct.

5. $\dfrac{s+1}{s+2}=0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of the equation by the denominator containing the unknown s:

$s+1=0$

The last operation on the unknown s is subtraction, specifically, $+1$. The inverse operation to $+1$ is -1. Apply this operation to both sides of the equation:

$s+1=0\ /-1$
$s=-1$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that solution is correct. Consider the LHS, substitute -1 for *s*,

$\dfrac{-1+1}{-1-1}=\dfrac{0}{-2}=0$.

Since LHS = RHS the solution is correct.

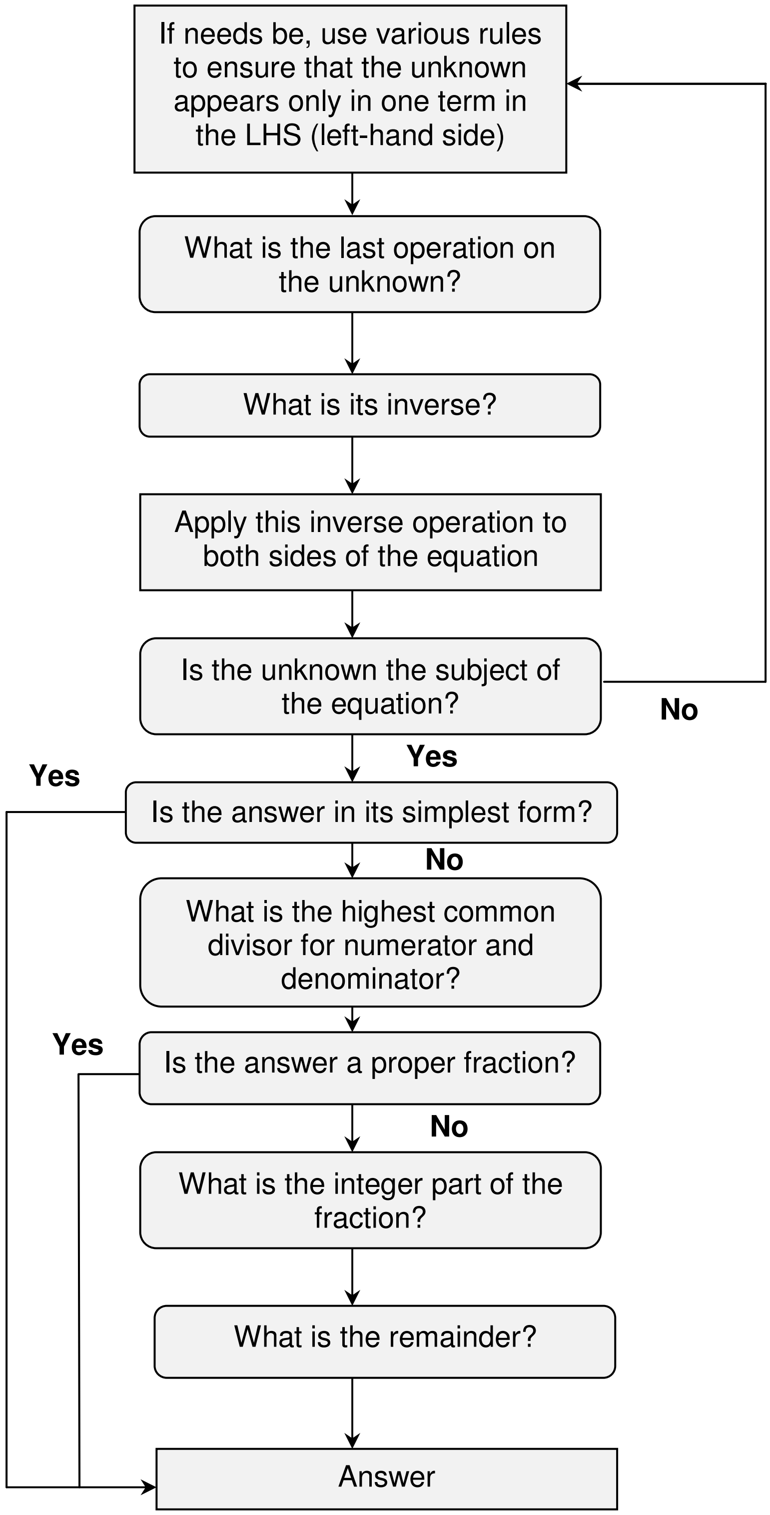


Figure 2.2. Decision Tree for Solving Simple Equations.

2.4 Applications of equations

Scientific and engineering systems are described using equations, sometimes referred to in science and engineering as **laws of nature, scientific formulae** or **principles** describing the relationships between various measurable quantities. If you have one equation (law, formula, principle), which contains one unknown, this equation (law, formula, principle) can be solved (**transposed, rearranged**) to find this unknown.

Example: A conker, of mass m=0.2 kg, falls vertically down from a tree in autumn. Whilst it falls it experiences the gravity force $F_g = mg$ and air resistance of magnitude $F_R = 0.4v$, where v is its speed in $m\ s^{-1}$. Calculate the speed at which it is falling when it has an acceleration $a = 1.81\ m\ s^{-2}$ (Take $g = 9.81\ m\ s^{-2}$).

Solution

Step 1. According to the 2nd Law of Newton,

$$F = ma$$

Step 2. The resulting force on the falling conker is

$$F = F_g - F_R \qquad (*)$$

Step 3. Using the statement of the problem, the right-hand side of the force balance equation (*) is

$$F_g - F_R = 0.2 \times 9.81 - 0.4v$$

Step 4. The left-hand side of the force balance equation (*) is

$$ma = 0.2 \times 1.81 = 3.62$$

Step 5. Thus, the force balance equation (*) for the falling conker can be written as

$$19.63 - 0.4v = 3.62$$

Step 6. Using the Decision Tree for Solving Simple Equations we can find the speed v.

2.5 A historical note

"Al-Khwarizmi (Mohammad ebne Mūsā Khwārazmī was a Persian mathematician, astronomer, astrologer and geographer. He was born around 780 in Khwārizm, then part of the Persian Empire (now Khiva, Uzbekistan) and died around 850. He worked most of his life as a scholar in the House of Wisdom in Baghdad. His *Algebra* was the first book on the systematic solution of linear and quadratic equations. Consequently he is considered to be the father of algebra, a title he shares with Diophantus. Latin translations of his *Arithmetic*, on the Indian numerals, introduced the decimal positional number system to the Western world in the twelfth century…

The word algebra is derived from *al-jabr*, one of the two operations used to solve quadratic equations, as described in his book… Al-ğabr is the process of … adding the same quantity to each side. For example, $x^2 = 40x - 4x^2$ is reduced to $5x^2 = 40x$."

http://en.wikipedia.org/wiki/Mu%E1%B8%A5ammad_ibn_M%C5%ABs%C4%81_al%E1%B8%B4w%C4%81rizm%C4%AB

2.6 Instructions for self-study

- **Revise Lecture 1 using the STUDY SKILLS Appendix**
- **Study Lecture 2 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Remove brackets

a) $(3 + x)(2 + y)$

b) $(x + 3)(x + 3)$

c) $(x + 3)(x - 3)$

Q2. Find the multiplicative inverse of $\dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$.

Q3. State and explain whether the following is correct:

a) $\dfrac{2 + 4ba}{2} = 1 + 2ab$

b) $\dfrac{2 + 4ab}{b} = 2 + 4a$

Q4. Factorise

a) $6x - 12xy$

b) $4x + 32z + 16y$

Q5. Solve

a) $5 - 2x = 2 + 3x$

b) $\dfrac{x+3}{2} = 3$

c) $\dfrac{3}{u+1} = \dfrac{2}{4u+1}$

d) $\dfrac{3}{t} = \dfrac{1}{2t+1}$

e) $\dfrac{u+1}{u-1} = \dfrac{1}{u-1}$

If you need more exercises you can use the following site:
http://www.mathtutor.ac.uk/viewdisks.php

Lecture 3. ALGEBRA: Exponentiation, Roots and Logarithms of Real Numbers

3.1 Types of variables and operations on variables (ctd.)

3.1.1 Operations: *n-th* power b^n

The symbol of the ***n-th* power** is a superscript n. Raising to power is also called **exponentiation**.

For whole numbers n,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ equal factors}}$$

Thus the symbol $(\ )^n$, where n is a whole number is a shorthand for a product of n equal factors.

Raising to power is a **direct operation** in the sense that we just define the result and raising a rational number to a whole power is a rational number.

Using the whole power indices m and n and multiplication rules the following useful rules can be verified:

Rules

1. $a^m \cdot a^n = a^{m+n}$ **product of powers with same base can be turned into a single power – just add indices**

2. $a^n \cdot b^n = (ab)^n$ **product of powers with the same index is power of products with this index**

3. $\dfrac{a^m}{a^n} = a^{m-n}$ **quotient of powers with same base can be turned into a single power – subtract indices**

4. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$ **quotient of powers is power of quotient**

5. $a^0 = 1$

6. $a^{-n} = \dfrac{1}{a^n}$

7. $(a^m)^n = a^{mn}$ **Convention:** $a^{m^n} = a^{(m^n)}$

3.1.2 Operations: *n-th* root

When n is the whole number the symbol of the ***n-th* root** is $\sqrt[n]{\ }$. The symbol of the square root is $\sqrt{\ }$. The n-*th* root, where n is a whole number is an **inverse operation** to raising to power n. This means that it can be defined via raising to power n:

Definition: $\sqrt[n]{b} = x: \quad x^n = b.$

The definition implies the following relations between raising to power n and the n-th root:

$\sqrt[n]{b^n} = b$ (taking the *n-th* root undoes raising to *n-th* power)
$(\sqrt[n]{b})^n = b$ (raising to *n-th* power undoes taking the *n-th* root)

Therefore, we can use notation $b^{1/n} = \sqrt[n]{b}$. Indeed, it is easy to check using the indices rules that exactly like the operation $\sqrt[n]{\ }$, the operation $(\,)^{1/n}$ undoes $(\,)^n$ and *vice versa.*

3.1.3 Variables: Irrational, Real and Complex Numbers

Extracting an n-th root is an operation inverse to raising to power n. It is the third inverse operation we encounter.

Question: what other inverse operations have we covered so far?
Answer:

As with subtraction and division, application of this new inverse operation might cause a difficulty:

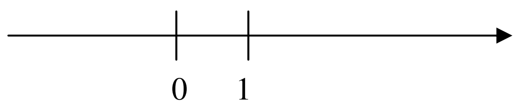
1. First of all, the answer might not be unique.

Question: what is a square root of $4, 9, 16 \ldots$?
Answer:

2. Also, extracting a whole root of a rational number does not always result in a rational number. However, it makes sense to say that extracting a whole root of a rational number introduces a new type of number called **irrational**, for example $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{3}$ *etc.* We get an idea of what an irrational number $\sqrt{2}$ is by saying that squaring it gives 2.

Question: what is an irrational number? Note that "ir" is Latin for "non".
Answer:

All rational numbers combined with all irrational numbers constitute a set of **real numbers.** A set of real numbers can be visualised as points on a number line. Any real number can be represented by a point on the number line and every point on the number line represents a real number (the proof of this lies outside the scope of these notes).



A **real variable** is a variable that takes real values.

The operation of taking a root also introduces **complex numbers** $\sqrt{-1}$ *etc.* Complex numbers are numbers that are not real and are to be discussed later. The main message: **an even root of a negative number cannot be a real number.**

Proof:

Indeed, **assume** that $\sqrt{-1} = x$, where x is real. Then squaring both sides, $-1 = x^2$. But any real number squared is positive or zero. Hence the **assumption** (that there exists a real number x such that $\sqrt{-1} = x$) **is invalid.**

A **complex variable** is a variable that takes complex values.

3.1.4 Operations: Logarithm Base b

The symbol of operation **taking a logarithm base *b*** is $\log_b$. Taking a logarithm base b is an **inverse operation** to raising b to power. Note that the base is always written as a subscript of $\log$. Since the logarithm is inverse to raising b to power it can be defined via this operation:

Definition: $\log_b a = n$: $b^n = a$.

The definition implies the following relations between raising b to power and log base b:

$\log_b(b^n) = n$ (taking $\log_b$ undoes taking b to power)
$b^{\log_b n} = n$ (taking b to power undoes $\log_b$)

We can use rules for indices to deduce rules for logs:

Rules

1. $\log_b xy = \log_b x + \log_b y$ **log of a product is sum of logs**
2. $\log_b \frac{x}{y} = \log_b x - \log_b y$ **log of a ratio is difference of logs**
3. $\log_b 1 = 0$ **log of 1 is 0**
4. $\log_b b = 1$
5. $\log_b \frac{1}{a} = -\log_b a$
6. $\log_b x^n = n \log_b x$ **log of a power is power times log**
7. $\log_b a = \frac{\log_c a}{\log_c b}$ **changing base rule**

3.1.5 Variables: Irrational, Real and Complex Numbers

The operation of taking a log is the fourth inverse operation we encounter.

Question: what other inverse operations have we covered so far?
Answer:

As with subtraction, division and taking the whole root, application of a log might cause a difficulty: Taking a log of a rational number does not always result in a rational number and taking a log of a real number does not always result in a real number. For example $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 5$ *etc.* are all irrational numbers.

Question: what is an irrational numbers? “Ir” stands for “non”.

Answer:

Also, $\log_{10}(-1)$ is not a real number. The main message: **if the base is positive the logarithm of a negative number cannot be a real number.**

General Comments

1. All laws and rules of addition, multiplication and taking to integer power are first introduced for whole numbers and can be verified (not proved) by substitution of whole numbers. It is then postulated (declared) that they extend to wider and wider number sets.

2. These operations can be applied to real numbers to produce results that are real too. Other algebraic operations when applied to real numbers do not necessarily produce results that are real.

3. We can now extend the Order of Operations rule to

Order of Operations (OOO)

1. Make implicit (invisible) brackets visible (everything raised and everything lowered with respect to the main line is considered to be bracketed)
2. Perform operations in brackets {[()]} first (inside out)

OOO →

Operations in Brackets	Powers	Multiplication	Addition
(including	powers	×	+
implicit)	roots	÷	-
	logs		

When dealing with expressions involving algebraic operations the convention is to perform operations in brackets first, then the most involved algebraic operations of power and its inverses, then an easier operation of multiplication and its inverse and finally the simplest algebraic operations of addition and its inverse.

3.2 Applications

3.2.1 Solving Quadratic Equations

The general form of a quadratic equation is

$$ax + bx+c=0$$

where constants a, b and c are assumed to be known and x is assumed to be an unknown variable.

Question: Why is it an equation?

Answer:

Question: Is it an algebraic equation? Why?
Answer:

Question: Is it a simple algebraic equation? Why?
Answer:

Question: What is a constant?
Answer:

We can solve any quadratic equation by using a trick called "completing the square".

Completing the square means re-writing the quadratic expression

$$ax^2+bx+c=(ax^2+bx)+c=a(x^2+\frac{b}{a}x)+c=a[(x+\frac{1}{2}\frac{b}{a})^2-(\frac{1}{2}\frac{b}{a})^2]+c=$$

$$a(x+\frac{b}{2a})^2-\frac{b^2}{4a}+c,$$

so that the unknown is present in only one term.

Any simple algebraic equation can be solved using the Decision Tree given in figure 2.1 and therefore completing the square can be used to prove that for any quadratic equation,

$$ax^2+bx+c=0$$

we have a formula for its solutions,

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The notation means that if we choose the top sign in the RHS, we get solution x_1 and if we choose the bottom sign, we get solution x_2. Let us now produce an (optional) proof of this formula:

Optional

Proof: Completing the square we can re-write the quadratic equation: $ax^2+bx+c=0$ as $a(x+\frac{b}{2a})^2-\frac{b^2}{4a}+c=0.$

We can now solve it using the Decision Tree for Solving Simple Equations.

Step 1: The last specific operation on x in the LHS is $-\frac{b^2}{4a}+c$.

Step 2: Its inverse is $+\frac{b^2}{4a}-c.$

Step 3: Applying this inverse operation to both sides of the equation we get

$$a(x+\frac{b}{2a})^2=\frac{b^2}{4a}-c$$

which can be re-written using the common denominator method as

$$a(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a}.$$

Step 4: x is not the subject of the equation, so we have to go through the solution loop again. The last specific operation on x in the LHS is $\times a$.

Step 5: Its inverse is $\times\frac{1}{a}$.

Step 6: Applying this inverse to both sides of the equation we get

$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

Step 7: x is not the subject of the equation, so we have to go through the solution loop again. The last operation on x in the LHS is squaring or raising to power 2.

Step 8: Its inverse is $\sqrt{\ }$, square root or $()^{1/2}$, power half.

Step 9: Applying this inverse to both sides of the equation we get

$$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$$

or

$$x_{1,2}+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

The notation means that if we choose the top sign on the right, we get solution x_1 and if we choose the bottom sign, we get solution x_2.

Step 10: The unknown is not the subject of the last equation, so we have to go through the solution loop again. The last specific operation on $x_{1,2}$ in the LHS is $+\frac{b}{2a}$.

Step 11: Its inverse is $-\frac{b}{2a}$.

Step 12: Applying this inverse to both sides of the equation, we get

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

3.2.2 Applications: Factorising Quadratic Expressions

Every quadratic expression $ax^2 + bx + c$ can be factorised as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where $x_{1,2}$ are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

solutions of the corresponding quadratic equation

$$ax^2 + bx + c = 0 .$$

$x_{1,2}$ are also known as **roots** of the quadratic.

You can check that factorisation makes sense:

Step 1. substitute x_1 into the LHS and then RHS. You get 0 on both sides.
Step 2. substitute x_2 into the LHS and then RHS. You get 0 on both sides.
Step 3. remove brackets in the RHS. The highest power term is ax^2. It is possible to get the other two terms too.

This factorisation (turning a quadratic sum into a product) proves extremely useful in many engineering and scientific applications.

3.3 Instructions for self-study

- **Revise Summaries on ALGEBRA and ORDER OF OPERATIONS**
- **Revise Lecture 1 and study Solutions to Exercises in Lecture 1 using the STUDY SKILLS Appendix**
- **Revise Lecture 2 using the STUDY SKILLS Appendix**
- **Study Lecture 3 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Turn into a single power

a) $2^3 \times 2^5$

b) $c^3 \times c^5$

c) $2^{2+j} \times 2^{2-j}$

d) $\dfrac{s^{1-\sqrt{2}}}{s^{2+\sqrt{2}}}$

e) $\dfrac{e^{1-j\sqrt{2}}}{e^{2+j\sqrt{2}}}$

Q2. Modify into an expression containing a power with a positive index

a) a^{-3}

b) u^{-1}

c) w^{-10}

Q3. Evaluate and check your result:

a) $\sqrt{4}$

b) $\sqrt{9}$

c) $\sqrt{16}$

d) $\sqrt{25}$

e) $36^{1/2}$

f) $49^{1/2}$

g) $8^{1/3}$

h) $27^{1/3}$

i) $\sqrt[3]{64}$

j) $\sqrt[3]{125}$

Q4. Evaluate without a calculator:

a) $\log_{10} 2 + \log_{10} 5$

b) $\log_{10} 200 - \log_{10} 2$. Check your result.

d) $\log_2 8$. Check your result $\log_{10} 10000$.

e) $\log_2 \frac{1}{2}$. Check your result.

f) $\log_{10} \frac{1}{100}$. Check your result.

Q5. Solve the following quadratic equations:

a) $s^2 + 5s + 6 = 0$

b) $x^2 + 4x + 4 = 0$

c) $s^2 - 36 = 0$

Q6. Factorise the following quadratic expressions:

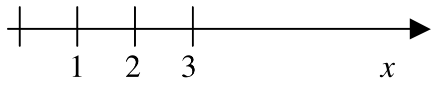
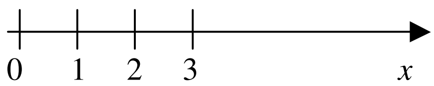
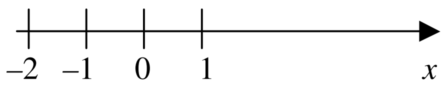
a) $s^2 + 5s + 6$

b) $x^2 + 4x + 4$

c) $s^2 - 36$

IV. SUMMARIES

Algebra Summary

OPERATIONS	TYPES OF VARIABLES
Addition (direct operation) Addition of whole numbers gives whole number 1. $a + b = b + a$ **Terminology:** a and b are called **terms** $a + b$ is called **sum** 2. $(a + b) + c = a + (b + c)$	**Whole numbers** are $1, 2, 3, \ldots$ 
Subtraction (inverse operation) **Def:** $a - b = x$: $x + b = a$ **Note:** $a + b - b = a$ **(subtraction undoes addition)** $a - b + b = a$ **(addition undoes subtraction)** 3. $a + 0 = a$ 4. for each a there exists one **additive inverse** $-a$: $a + (-a) = 0$ **Rules** (follow from Laws): $+ (b + c) = + b + c$ $+ \; a + b = a + b$ $- (- a) = a$ $- (a) = -a$	introduces **0** and **negative numbers**: $a - a = 0$ if $b > a$ $\quad a - b = - (b - a)$ **Natural numbers** are $0, 1, 2, \ldots$  **Integers** are $\ldots, -2, -1, 0, 1, 2, \ldots$ 

Multiplication (direct operation)

For whole numbers n

$$a\cdot n = \underbrace{a+\ldots\ldots\ldots\ldots+a}_{n \text{ terms}}$$

Notation: $ab = a\cdot b = a\times b$

$2b = 2\cdot b = 2\times b$

$23 \neq 2\cdot 3,\ 23 = 2\cdot 10+3$

$2\frac{1}{2} \neq 2\cdot\frac{1}{2},\ 2\frac{1}{2} = 2+\frac{1}{2}$

$2\frac{3}{2} = 2\cdot\frac{3}{2}$

1. $a\cdot b = a\cdot b$

Terminology: a and b are called **factors**

ab - **product**

2. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

Conventions: $abc = (ab)c$

$a(-bc) = -abc$

3. $a(b+c) = ab+ac$

⟶ Removing brackets

⟵ Factoring

4. $a\cdot 0 = 0$
5. $a\cdot 1 = a$

Rules (follow from Laws):

$(a+b)(c+d) = ac+ad+bc+bd$ (**SMILE RULE**)

$(-1)\cdot n = -n$

$(-1)\cdot(-1) = 1$

Division (inverse operation)

Def: $a/b = x$: $xb = a$

Terminology: a – **numerator**

b – **denominator**

a/b – **fraction (ratio)**

- **proper fraction** if $|a|<|b|$, a, b integers

Note: $ab/b = a$ **(division undoes multiplication)**

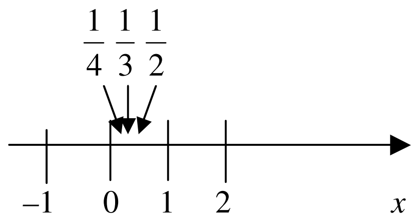
$(a/b)b = a$ **(multiplication undoes division)**

6. For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $a\cdot 1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\frac{m}{n}$,

where m and $n \neq 0$ are integers (**division by zero is not defined**)



Rules:

$$\frac{a}{b}\cdot n = \frac{an}{b}$$

$$\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$$

$$\frac{an}{bn} = \frac{a\cdot \not{n}}{b\cdot \not{n}} = \frac{a}{b}$$ **CANCELLATION**

$$\frac{\frac{1}{n}}{m} = \frac{m}{n}$$ **FLIP RULE**

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$ (with d and b crossed out above a and c)

Note: $\frac{a+c}{b} = (a+c)/b$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd}$$ **COMMON DENOMINATOR RULE**

n-th **power** b^n (direct operation)

For whole numbers n

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}$$

Rules:

$a^m \cdot a^n = a^{m+n}$

(product of powers with the same base is a power with indices added)

$a^n \cdot b^n = (ab)^n$

(product of powers is power of product)

$\frac{a^m}{a^n} = a^{m-n}$

(ratio of powers is power of ratios)

$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$

(ratio of powers with the same base – subtract indices)

$a^0 = 1$

$a^{-n} = \frac{1}{a^n}$

$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

n-th **root** (inverse to taking to power n)

Def: $\sqrt[n]{b} = x: \ \mathrm{x}^n = b$

Note:

$\sqrt[n]{b^n} = b$

(taking *n-th* root undoes taking *n-th* power)

$(\sqrt[n]{b})^n = b$

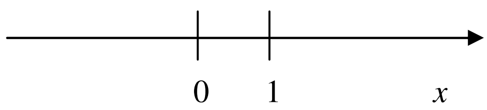
(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = \sqrt[n]{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$ …

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole number line (called **real line** for this reason)



Logarithm base b (inverse to taking b to power)

Def: $\log_b a = n: \; b^n = a$

Note:
$\log_b b^n = n$ (check using definition: $b^n = b^n$)
(taking $\log_b$ undoes taking b to power)

$b^{\log_b n} = n$

(**taking b to power undoes $\log_b$**)

Rules (follow from Rules for Indices):
$\log_b xy = \log_b x + \log_b y$
(log of a product is sum of logs)

$\log_b \frac{x}{y} = \log_b x - \log_b y$
(log of a ratio is difference of logs)
$\log_b 1 = 0$ **(log of 1 is 0)**
$\log_b b = 1$
$\log_b \frac{1}{a} = -\log_b a$
$\log_b x^n = n \log_b x$
(log of a power is power times log)

$\log_b a = \frac{\log_c a}{\log_c b}$
(changing base)

introduces **irrational** (not rational**)** **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers,** $\sqrt{-1}, \log_{10}(-1)$, *etc.*

General remarks

1. $a - b = a + (-b)$ $\longrightarrow$ a difference can be re-written as a sum

2. $\frac{a}{b} = a \cdot \frac{1}{b} = ab^{-1}$ $\longrightarrow$ a ratio can be re-written as a product

3. $\sqrt[n]{b} = b^{1/n}$ $\longrightarrow$ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

Functions Summary

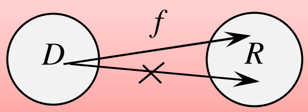
Variables are denoted mostly by $x, y, z, p, \ldots\ldots w$.
A variable can take any value from a set of allowed numbers.

Functions are denoted mostly by f, g and h or $f\,(\,)$, $g\,(\,)$ and $h(\,)$
(no multiplication sign is intended!).

In mathematics the word *function* has three meanings:

1. $f(\,)$ - **an operation or a chain of operations** on an independent variable;
2. $f(x)$ - a **dependent variable**, that is the variable obtained when $f(\,)$ acts on an independent variable x;
3. Set of pairs $\{(x, f(x)$: $f(\,)$ assigns one value $f(x)$ to every allowed value of $x\}$.

A diagrammatical representation of a function



To specify a function we need to specify a **rule** - (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, $f\,(x)$ assigns <u>one and only one</u> value $y \in R$ (**range**, the set of all possible values of the dependent variable).

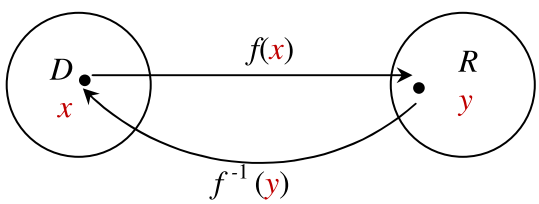
Inverse functions

$f^{-1}(x)$: $f \circ f^{-1}(x) = f^{-1} \circ f\,(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function



$y = f(x)$

$x = f^{-1}(y)$

Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. By convention, the **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**.

Other **operations** must be performed in the order of decreasing complexity, which is

oiB - **operations in Brackets**

P- Powers (including inverse operations of roots and logs)

M - Multiplication (including inverse operation of division)

A - Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of operations (OOO)

1. Make **implicit** (invisible) brackets visible (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO →

oiB	P	M	A
(including implicit)	$(\ )^{(\ )}$	$\times$	+
	roots	$\div$	-
	logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2 + bx + c$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared): b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2 + bx + c = 0$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be 360° or 2π (radians)

$$\Rightarrow 2\pi\ (rad) = 360^0$$
$$\Rightarrow 1\ (rad) \approx 57^0$$

Above, x is the number of **radians** (angles $\approx 57^0$) in a given angle (*cf*: $x\,(\text{m}) = x \cdot 1\text{m}$, so that x is the number of **units of length** (segments 1m long) in a given segment)

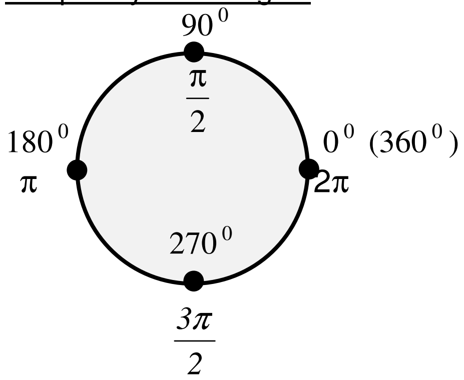
(**Note**: ***cf.*** means ***compare***)

The radian is a dimensionless unit of angle.

$$\Rightarrow\ x\ (rad) = x\frac{180^0}{\pi(rad)} = y^0,\ \ y^0 = y\frac{\pi(rad)}{180^0} = x\ (rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

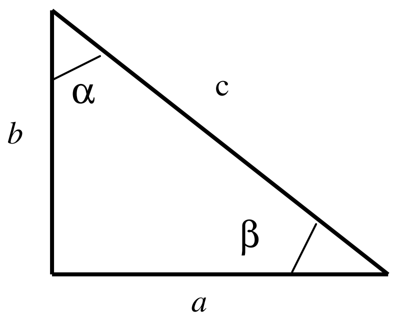
Frequently used angles



$30^0 = (\frac{30\pi}{180} =)\ \frac{\pi}{6}$ $60^0 = (\frac{60\pi}{180} =)\ \frac{\pi}{3}$ $120^0 = (\frac{120\pi}{180} =)\ \frac{2\pi}{3}$	$45^0 = \frac{\pi}{4}$

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90^0) as follows:



$$\sin\alpha = \cos\beta = \frac{a}{c}$$

$$\cos\alpha = \sin\beta = \frac{b}{c}$$

$$\tan a = \cot\beta = \frac{a}{b}$$

$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin\frac{\pi}{6} = \frac{1}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
	$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

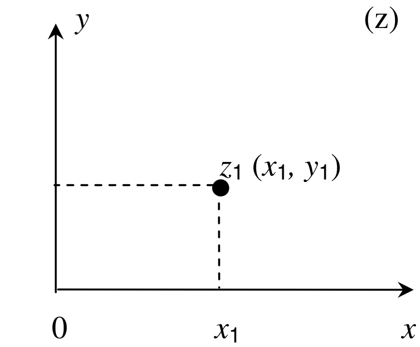
Trigonometric identities

$\sin^2 x + \cos^2 x = 1$
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos 2x = \cos^2 x - \sin^2 x$
$\sin 2x = 2\sin x \cos x$

$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x+\alpha)$, where $\tan\alpha = \frac{A}{B}$

Complex Numbers

The Cartesian representation of a complex number

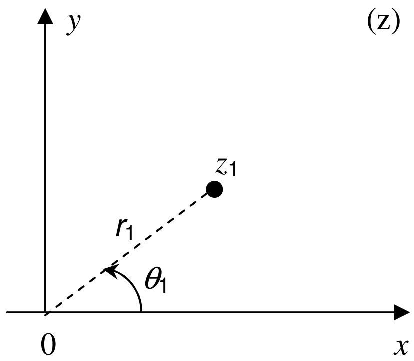


real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z = x + jy$, where x, y – real, $j = \sqrt{-1}$

x-coordinate y-coordinate (on the ordinary plane)

The exponential representation of a complex number



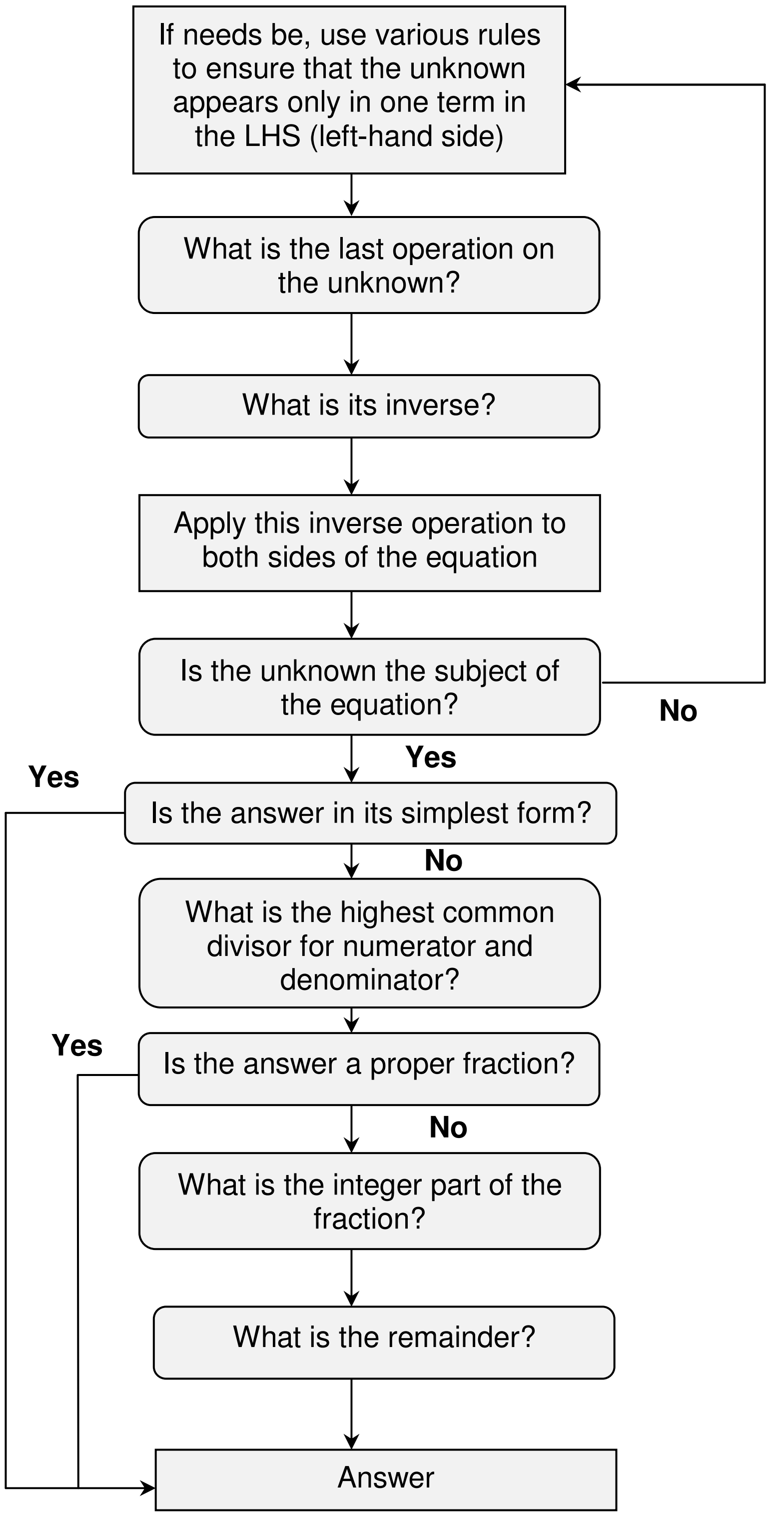
magnitude, $|z|$ **argument**, $\arg(z)$ (on the Argand diagram)

$z = r\, e^{j\theta}$, where $r \geq 0$, θ - angle (in degrees or radians)

polar radius r (distance to the origin)

polar angle θ (on the ordinary plane)

Decision Tree For Solving Simple Equations



V. GLOSSARY

ABSTRACTION - a general concept formed by extracting common features from specific examples.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by a common divisor.

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

CONCEPT - a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN.

If there is only one CONSTANT, the preferred choice for its symbol is a. The second choice is b, and the third, c, so e.g. in the EXPRESSION $x + a$, x is usually understood to represent a VARIABLE and a, a constant.

If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM - a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s,* kg/m^3, £.

EQUATION – a mathematical statement involving UNKNOWNS and the = sign which can be true or false, e.g. $2x + 3y = 10$ is true when $x = y = 2$ and not true when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. $2(a + b)$, $2ab$.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, a and b.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

GENERALISATION - an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH - a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INTEGER PART – when dividing a positive integer m into a positive integer n, k is the INTEGER PART if it is the largest positive integer producing $k*m \leq n$. The REMAINDER is the difference $n - k*m$, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that undoes what the original OPERATION does.

LAST OPERATION – see ORDER OF OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION –– an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. $2x - 3 = 0$ is a LINEAR EQUATION, 2 is a COEFFICIENT in front of the UNKNOWN and -3 is a FREE TERM.

NON-DIMENSIONAL QUANTITY - a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. $2 \ln (x) - 3 = 0$ is a NON-LINEAR EQUATION N in x.

OPERATION (mathematical) – action on CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which to perform the OPERATIONS. By convention, the

ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for is fractions: when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent. Other OPERATIONS must be performed in order of decreasing complexity, which is

POWERS (including inverse operations of roots and logs)

MULTIPLICATION (including inverse operation of division)

ADDITION (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, e.g. ab is a PRODUCT, and so is $(a + b)c$.

QUOTIENT - a mathematical expression where the last operation is division.

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the = sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SIMPLE EQUATION – an EQUATION that can be rearranged to contain the unknown in one term only.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT - A is a SUFFICIENT condition of B if $A \Rightarrow B$ (A implies B), so that if A is satisfied, then B takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see ORDER OF OPERATIONS) is addition, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value or expression can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its symbol is x. The second choice is y and the third, z. If there are more variables, then one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

VI. STUDY SKILLS FOR MATHS

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours each week**:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.

2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**

3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).

4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

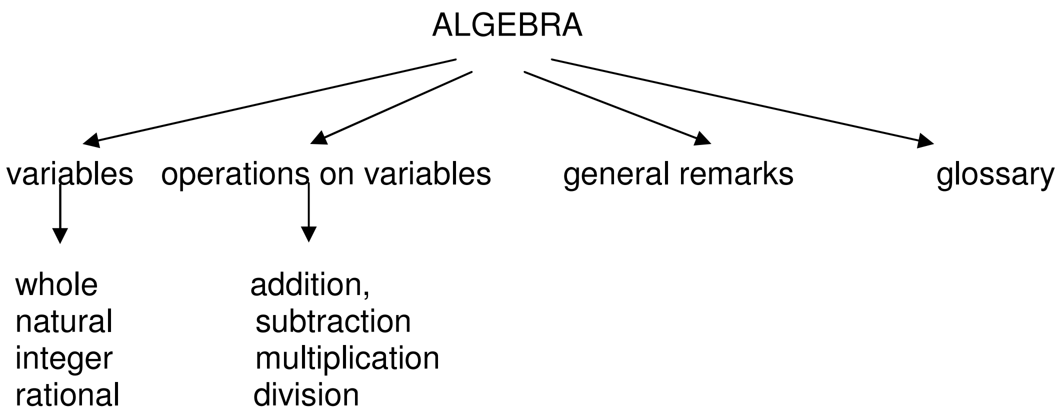
Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

Thus, you can construct the flow chart which looks like that



2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.

3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing.** Try this a few times before checking with the notes which step is a problem.

5. Repeat the process for each problem.

6. Repeat the process for each subtopic.

7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.

2. Keep consulting the **Glossary**.

3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.

4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to “make invisible brackets visible”.

2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.

3. The concept of **inverse operation**.

4. The **Decision Tree for Solving Simple Equations**.

5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.

6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real function of real variable (real argument). You should know the precise meaning of the word constant (you should always say "constant with respect to (the independent variable) x or t or whatever..."

7. How to do function composition and decomposition using **Order of Operations**

8. How to use graphs

9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos

10. The **Trigonometry Summary**

11. The approximate values of e (≈ 2.71) and π (≈ 3.14)

12. What is j ($= \sqrt{-1}$) and what is j^2 ($=-1$)

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand diagram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

15. The **Sketching by Simple Transformation Summary**

VII. TEACHING METHODOLOGY (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question - Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft’s book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGIG AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do an exercise yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) - rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education - development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you - in its essence - for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise a sequence of names, facts or dates there are well established techniques promoted in various books on memory. They suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with these objects. However, this technique will not work with technical information. What you need to establish are much deeper - meaningful - connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now - in their full glory - LATER, in year 2 and 3, not to-day.

Hope this helps!